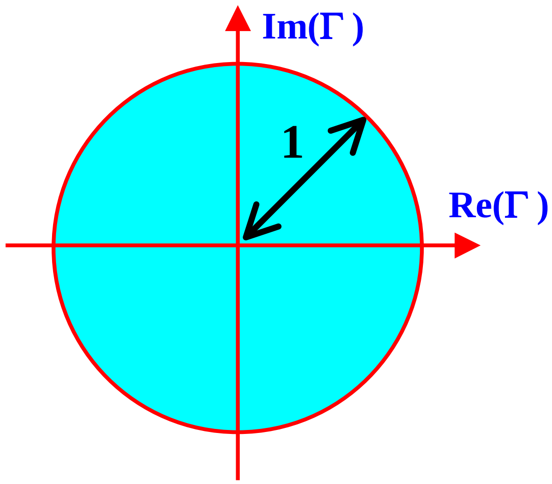
Smith Chart

The Smith chart is one of the most useful graphical tools for high frequency circuit applications. The chart provides a clever way to visualize complex functions and it continues to endure popularity decades after its original conception.

From a mathematical point of view, the Smith chart is simply a representation of all possible complex impedances with respect to coordinates defined by the reflection coefficient.



The domain of definition of the reflection coefficient is a circle of radius 1 in the complex plane. This is also the domain of the Smith chart.

The goal of the Smith chart is to identify all possible impedances on the domain of existence of the reflection coefficient. To do so, we start from the general definition of line impedance (which is equally applicable to the load impedance)

$$Z(d) = \frac{V(d)}{I(d)} = Z_0 \frac{1+\Gamma(d)}{1-\Gamma(d)}$$

This provides the complex function $Z(d) = f\{\mathrm{Re}(\Gamma), \mathrm{Im}(\Gamma)\}$ that we want to graph. It is obvious that the result would be applicable only to lines with exactly characteristic impedance Z_0.

In order to obtain universal curves, we introduce the concept of normalized impedance

$$z(d) = \frac{Z(d)}{Z_0} = \frac{1+\Gamma(d)}{1-\Gamma(d)}$$

The normalized impedance is represented on the Smith chart by using families of curves that identify the normalized resistance *r* (real part) and the normalized reactance *x* (imaginary part)

$$z(d) = \mathrm{Re}(z) + j\,\mathrm{Im}(z) = r + jx$$

Let's represent the reflection coefficient in terms of its coordinates

$$\Gamma(d) = \mathrm{Re}(\Gamma) + j\,\mathrm{Im}(\Gamma)$$

Now we can write

$$r + jx = \frac{1 + \mathrm{Re}(\Gamma) + j\,\mathrm{Im}(\Gamma)}{1 - \mathrm{Re}(\Gamma) - j\,\mathrm{Im}(\Gamma)}$$

$$= \frac{1 - \mathrm{Re}^2(\Gamma) - \mathrm{Im}^2(\Gamma) + j2\,\mathrm{Im}(\Gamma)}{(1 - \mathrm{Re}(\Gamma))^2 + \mathrm{Im}^2(\Gamma)}$$

The real part gives

Add a quantity equal to zero

$$r = \frac{1 - \mathrm{Re}^2(\Gamma) - \mathrm{Im}^2(\Gamma)}{\left(1 - \mathrm{Re}(\Gamma)\right)^2 + \mathrm{Im}^2(\Gamma)}$$

$$r\left(\mathrm{Re}(\Gamma) - 1\right)^2 + \left(\mathrm{Re}^2(\Gamma) - 1\right) + r\,\mathrm{Im}^2(\Gamma) + \mathrm{Im}^2(\Gamma) + \overbrace{\frac{1}{1+r} - \frac{1}{1+r}}^{=\,0} = 0$$

$$\left[r\left(\mathrm{Re}(\Gamma) - 1\right)^2 + \left(\mathrm{Re}^2(\Gamma) - 1\right) + \frac{1}{1+r}\right] + (1+r)\,\mathrm{Im}^2(\Gamma) = \frac{1}{1+r}$$

$$(1+r)\left[\mathrm{Re}^2(\Gamma) - 2\,\mathrm{Re}(\Gamma)\frac{r}{1+r} + \frac{r^2}{(1+r)^2}\right] + (1+r)\,\mathrm{Im}^2(\Gamma) = \frac{1}{1+r}$$

$$\Rightarrow \qquad \left[\mathrm{Re}(\Gamma) - \frac{r}{1+r}\right]^2 + \mathrm{Im}^2(\Gamma) = \left(\frac{1}{1+r}\right)^2$$

Equation of a circle

The imaginary part gives

Multiply by x and add a quantity equal to zero

$$x = \frac{2\,\mathrm{Im}(\Gamma)}{(1-\mathrm{Re}(\Gamma))^2 + \mathrm{Im}^2(\Gamma)}$$

$$x^2\left[(1-\mathrm{Re}(\Gamma))^2 + \mathrm{Im}^2(\Gamma)\right] - 2x\,\mathrm{Im}(\Gamma) + \underbrace{1-1}_{=0} = 0$$

$$\left[(1-\mathrm{Re}(\Gamma))^2 + \mathrm{Im}^2(\Gamma)\right] - \frac{2}{x}\mathrm{Im}(\Gamma) + \frac{1}{x^2} = \frac{1}{x^2}$$

$$(1-\mathrm{Re}(\Gamma))^2 + \left[\mathrm{Im}^2(\Gamma) - \frac{2}{x}\mathrm{Im}(\Gamma) + \frac{1}{x^2}\right] = \frac{1}{x^2}$$

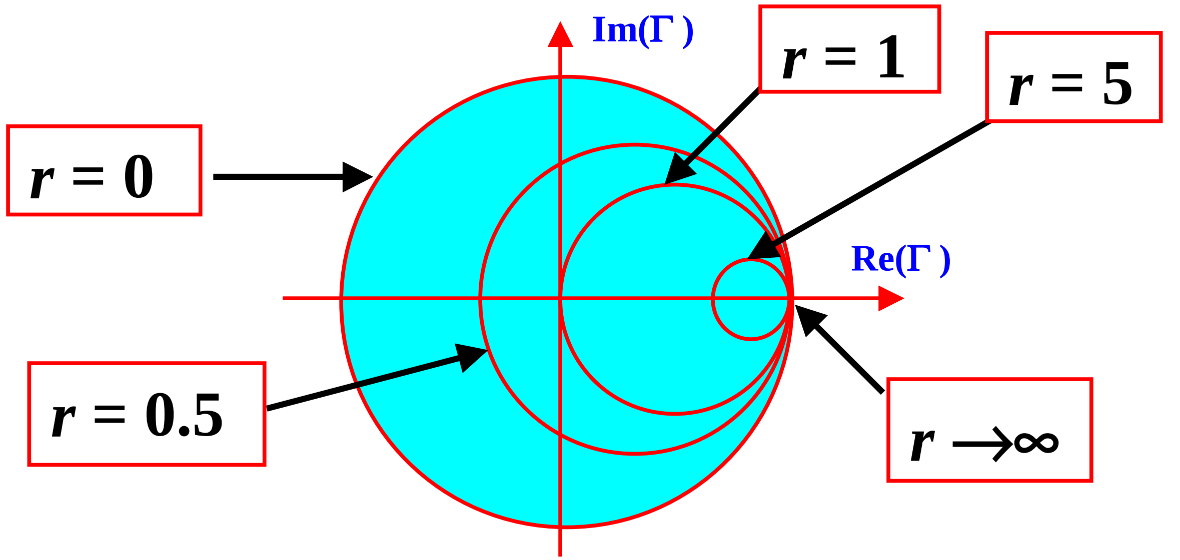
$$\Rightarrow \qquad (\mathrm{Re}(\Gamma)-1)^2 + \left[\mathrm{Im}(\Gamma) - \frac{1}{x}\right]^2 = \frac{1}{x^2}$$

Equation of a circle

The result for the real part indicates that on the complex plane with coordinates (Re(Γ), Im(Γ)) all the possible impedances with a given normalized resistance r are found on a circle with

$$\text{Center} = \left\{ \frac{r}{1+r}, 0 \right\} \qquad \text{Radius} = \frac{1}{1+r}$$

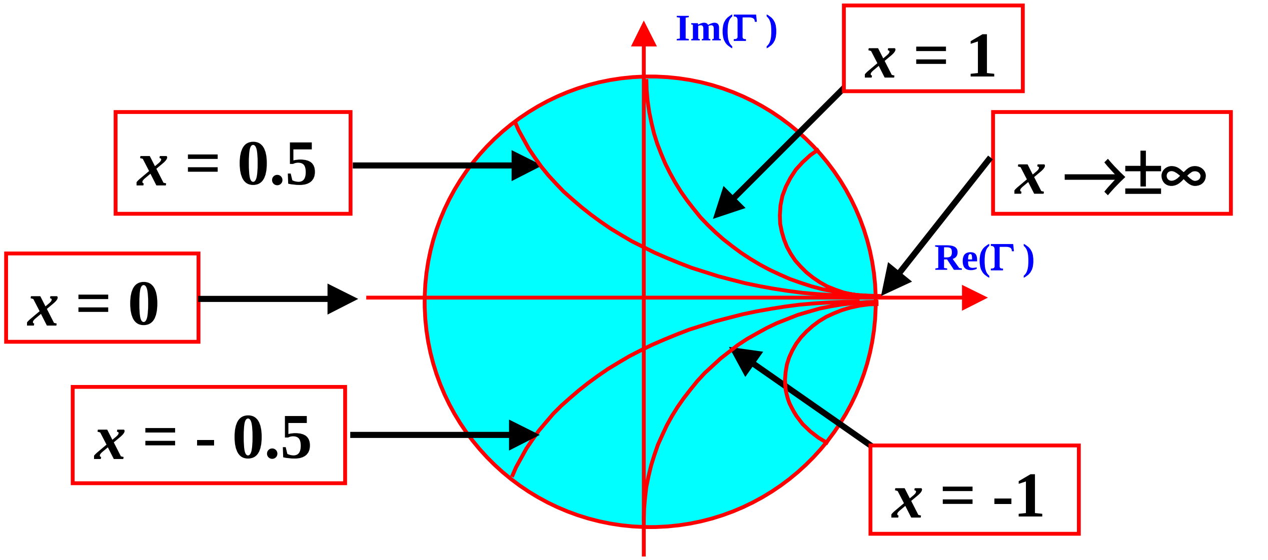
As the normalized resistance r varies from 0 to ∞, we obtain a family of circles completely contained inside the domain of the reflection coefficient $|\,\Gamma\,| \leq 1$.



The result for the imaginary part indicates that on the complex plane with coordinates (Re(Γ), Im(Γ)) all the possible impedances with a given normalized reactance x are found on a circle with

$$\text{Center} = \left\{ 1\,,\ \frac{1}{x} \right\} \qquad\qquad \text{Radius} = \frac{1}{x}$$

As the normalized reactance x varies from -∞ to ∞ , we obtain a family of arcs contained inside the domain of the reflection coefficient $|\,\Gamma\,| \leq 1$.



Basic Smith Chart techniques for loss-less transmission lines

- **Given $Z(\mathrm{d}) \Rightarrow$ Find $\Gamma(\mathrm{d})$**
  Given $\Gamma(\mathrm{d}) \Rightarrow$ Find $Z(\mathrm{d})$

- **Given Γ_R and $Z_R \Rightarrow$ Find $\Gamma(\mathrm{d})$ and $Z(\mathrm{d})$**
  Given $\Gamma(\mathrm{d})$ and $Z(\mathrm{d}) \Rightarrow$ Find Γ_R and Z_R

- **Find d_{max} and d_{min} (maximum and minimum locations for the voltage standing wave pattern)**

- **Find the Voltage Standing Wave Ratio (VSWR)**

- **Given $Z(\mathrm{d}) \Rightarrow$ Find $Y(\mathrm{d})$**
  Given $Y(\mathrm{d}) \Rightarrow$ Find $Z(\mathrm{d})$

Given $Z(\mathrm{d})$ $\Rightarrow$ Find $\Gamma(\mathrm{d})$

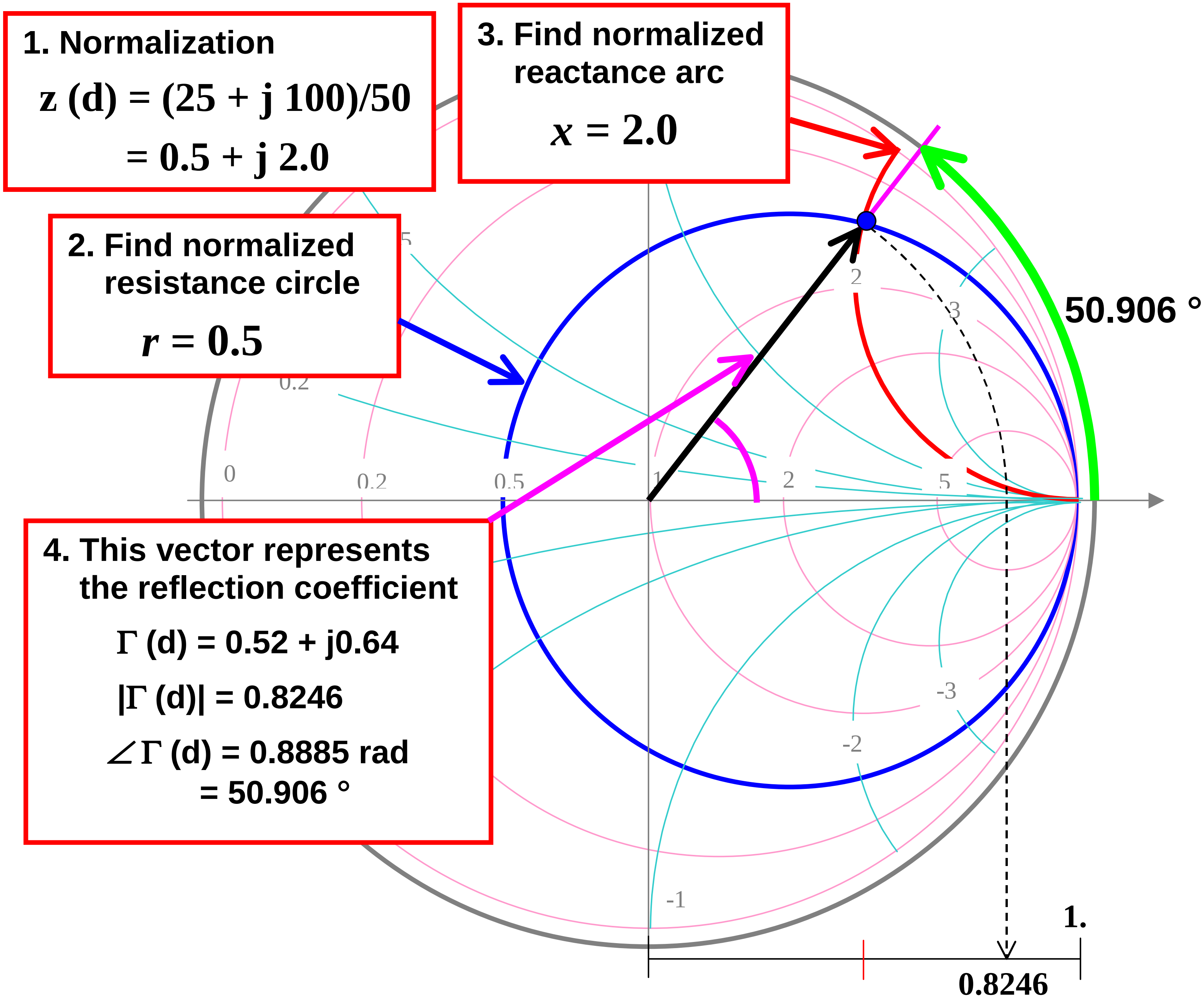
1. **Normalize the impedance**

$$z(\mathrm{d}) = \frac{Z(\mathrm{d})}{Z_0} = \frac{R}{Z_0} + j\,\frac{X}{Z_0} = r + j\,x$$

2. **Find the circle of constant normalized resistance r**
3. **Find the arc of constant normalized reactance x**
4. **The intersection of the two curves indicates the reflection coefficient in the complex plane. The chart provides directly the magnitude and the phase angle of $\Gamma(\mathrm{d})$**

Example: Find $\Gamma(\mathrm{d})$, given

$$Z(\mathrm{d}) = 25 + j\,100\ \Omega \qquad \text{with} \quad Z_0 = 50\ \Omega$$


1. Normalization
z (d) = (25 + j 100)/50
= 0.5 + j 2.0
2. Find normalized resistance circle
r = 0.5
3. Find normalized reactance arc
x = 2.0
4. This vector represents the reflection coefficient
Γ (d) = 0.52 + j0.64
|Γ (d)| = 0.8246
∠Γ (d) = 0.8885 rad
= 50.906 °
50.906 °
1.
0.8246
0
0.2
0.5
1
2
5
-1
-2
-3
3

Given $\Gamma(\mathrm{d}) \Rightarrow$ Find $Z(\mathrm{d})$

1. **Determine the complex point representing the given reflection coefficient $\Gamma(\mathrm{d})$ on the chart.**
2. **Read the values of the normalized resistance r and of the normalized reactance x that correspond to the reflection coefficient point.**
3. **The normalized impedance is**

$$z(\mathrm{d}) = r + j\,x$$

and the actual impedance is

$$Z(\mathrm{d}) = Z_0\; z(\mathrm{d}) = Z_0\,(r + j\;x) = Z_0\;r + j\;Z_0\;x$$

Given Γ_R and Z_R $\Longleftrightarrow$ Find $\Gamma(\mathrm{d})$ and $Z(\mathrm{d})$

NOTE: the magnitude of the reflection coefficient is constant along a loss-less transmission line terminated by a specified load, since

$$\left|\Gamma(\mathrm{d})\right| = \left|\Gamma_R \exp(-j2\beta \mathrm{d})\right| = \left|\Gamma_R\right|$$

Therefore, on the complex plane, a circle with center at the origin and radius $|\Gamma_R|$ represents all possible reflection coefficients found along the transmission line. When the circle of constant magnitude of the reflection coefficient is drawn on the Smith chart, one can determine the values of the line impedance at any location.

The graphical step-by-step procedure is:

1. **Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.**

2. **Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.**
3. **Starting from the point representing the load, travel on the circle in the clockwise direction, by an angle**

$$\theta = 2\,\beta\,d = 2\,\frac{2\pi}{\lambda}\,d$$

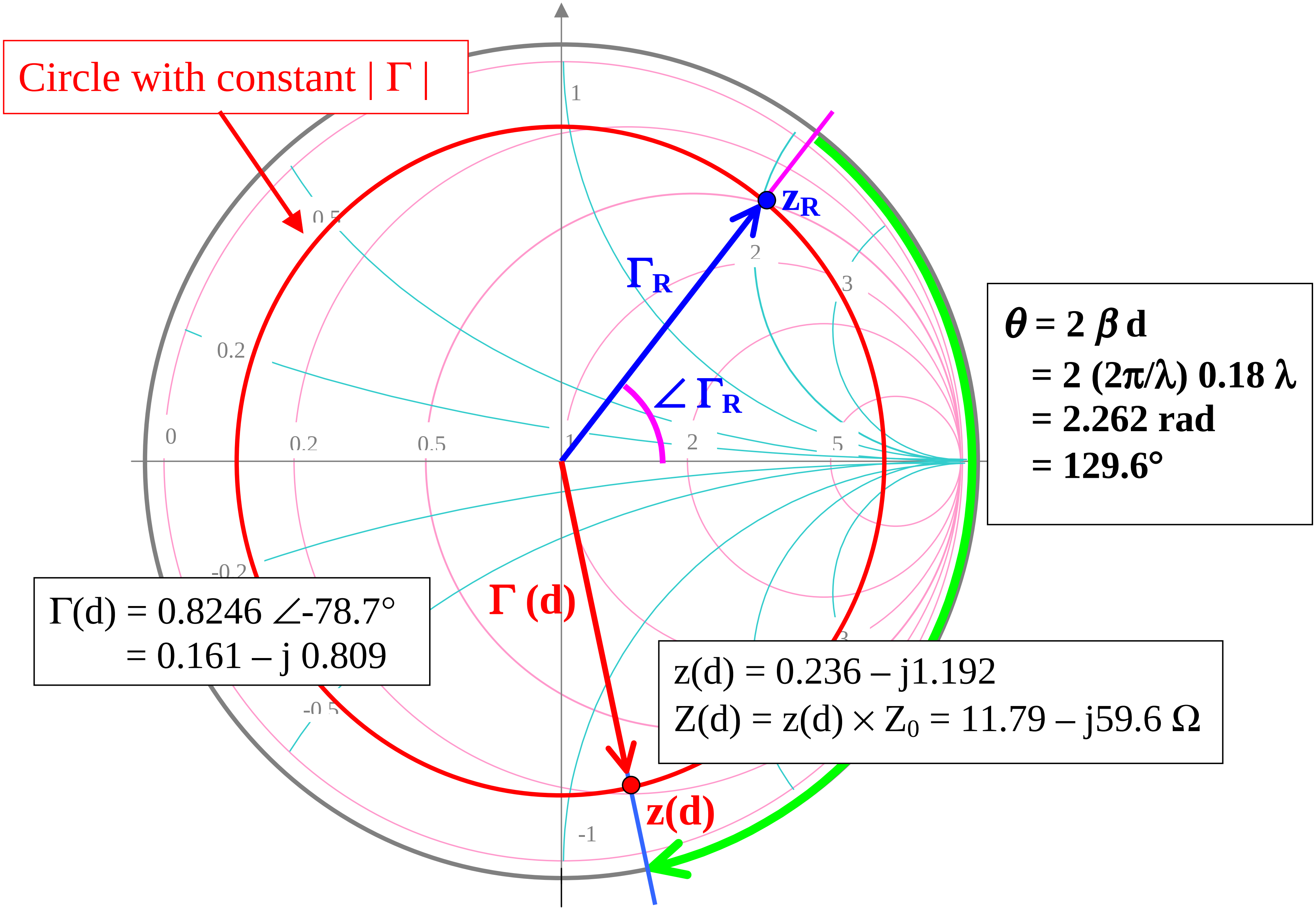
4. **The new location on the chart corresponds to location d on the transmission line. Here, the values of $\Gamma(d)$ and $Z(d)$ can be read from the chart as before.**

Example: Given

$$Z_R = 25 + j\,100\;\Omega \qquad \text{with} \qquad Z_0 = 50\;\Omega$$

find

$$Z(d) \quad \text{and} \quad \Gamma(d) \qquad \text{for} \qquad d = 0.18\lambda$$


Circle with constant | Γ |
z_R
Γ_R
$\angle \Gamma_R$
Γ (d)
z(d)
$\boldsymbol{\theta} = 2\,\boldsymbol{\beta}\,\mathbf{d}$
$= 2\,(2\pi/\lambda)\;0.18\;\lambda$
$= 2.262$ rad
$= 129.6°$
$\Gamma(d) = 0.8246\ \angle{-78.7°}$
$= 0.161 - j\,0.809$
$z(d) = 0.236 - j1.192$
$Z(d) = z(d) \times Z_0 = 11.79 - j59.6\ \Omega$
1
0.5
0.2
0
0.2
0.5
1
2
5
2
3
-0.2
-0.5
-1

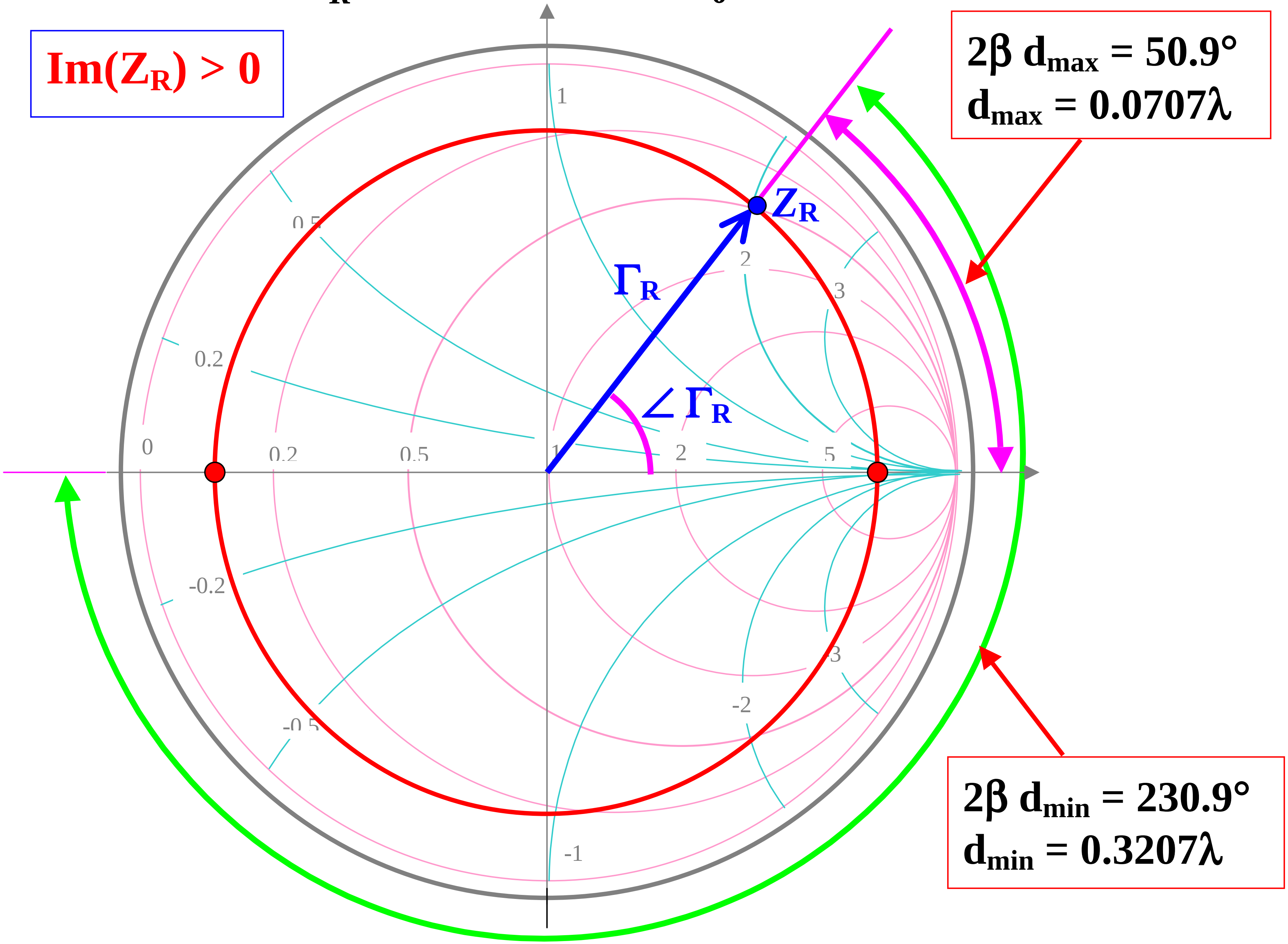
Given Γ_R and Z_R $\Rightarrow$ Find d_{max} and d_{min}

1. **Identify on the Smith chart the load reflection coefficient Γ_R or the normalized load impedance Z_R .**
2. **Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$. The circle intersects the real axis of the reflection coefficient at two points which identify d_{max} (when $\Gamma(d)$ = Real positive) and d_{min} (when $\Gamma(d)$ = Real negative)**
3. **A commercial Smith chart provides an outer graduation where the distances normalized to the wavelength can be read directly. The angles, between the vector Γ_R and the real axis, also provide a way to compute d_{max} and d_{min} .**

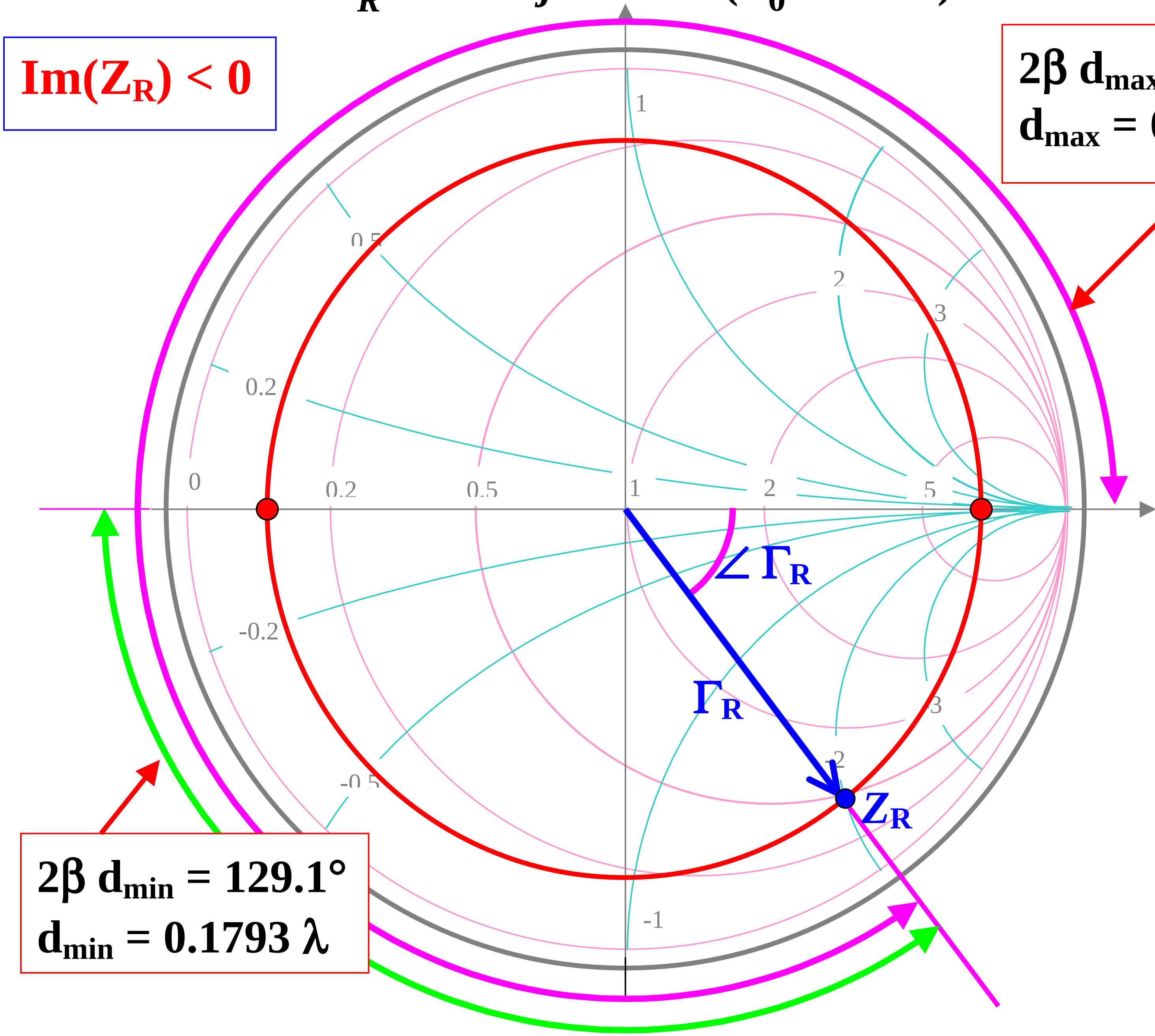
Example: Find d_{max} and d_{min} for

$$Z_R = 25 + j\,100\,\Omega \;\; ; \;\; Z_R = 25 - j100\Omega \qquad (Z_0 = 50\,\Omega)$$

$$Z_R = 25 + j\,100\ \Omega \quad (Z_0 = 50\ \Omega)$$



$$Z_R = 25 - j\,100\ \Omega \quad (Z_0 = 50\ \Omega)$$



Given Γ_R and Z_R $\Rightarrow$ Find the Voltage Standing Wave Ratio (VSWR)

The Voltage standing Wave Ratio or VSWR is defined as

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma_R|}{1-|\Gamma_R|}$$

The normalized impedance at a maximum location of the standing wave pattern is given by

$$z(d_{\max}) = \frac{1+\Gamma(d_{\max})}{1-\Gamma(d_{\max})} = \frac{1+|\Gamma_R|}{1-|\Gamma_R|} = VSWR!!!$$

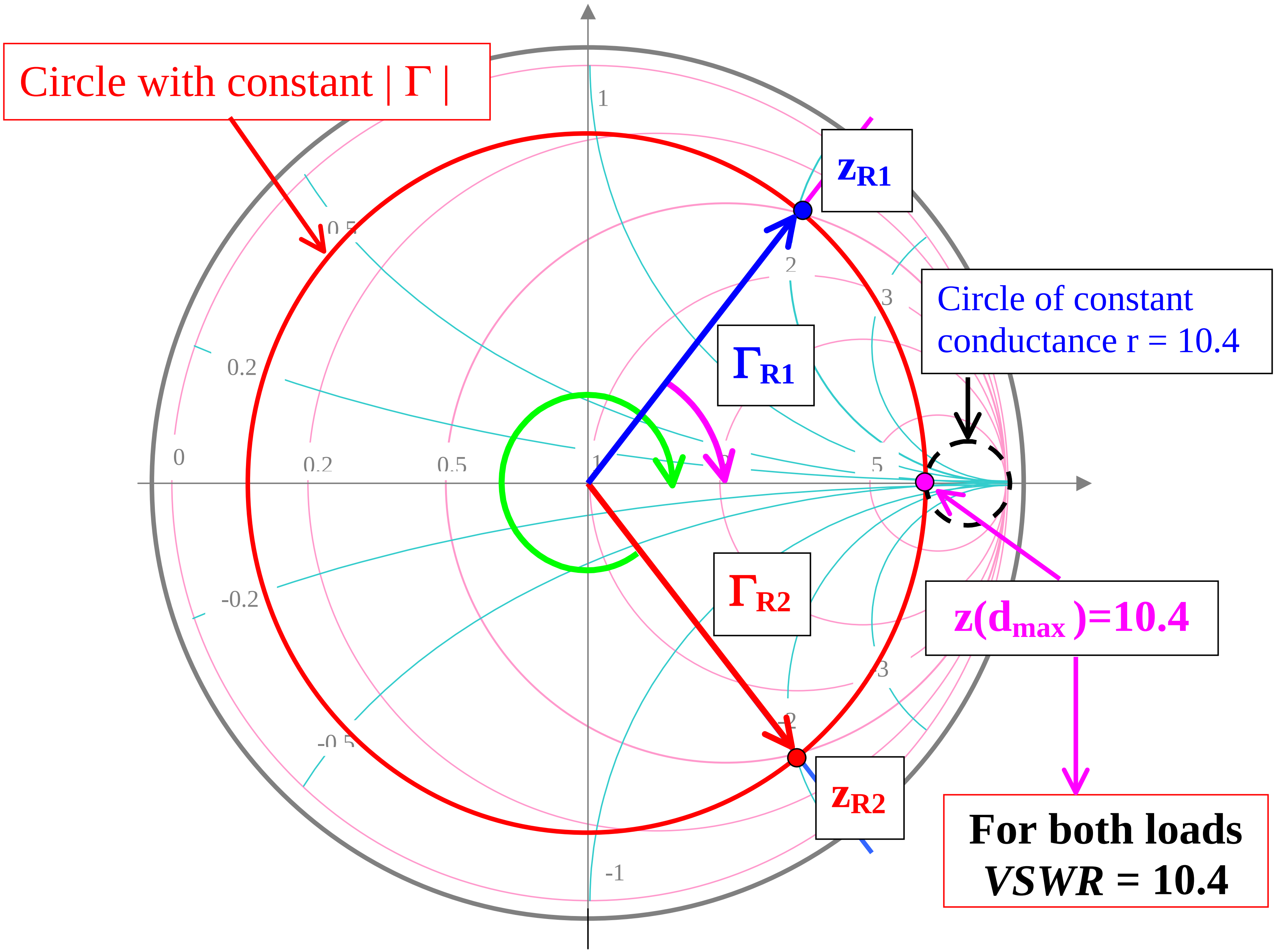
This quantity is always real and ≥ 1. The VSWR is simply obtained on the Smith chart, by reading the value of the (real) normalized impedance, at the location $d_{\max}$ where Γ is real and positive.

The graphical step-by-step procedure is:

1. **Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.**
2. **Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.**
3. **Find the intersection of this circle with the real positive axis for the reflection coefficient (corresponding to the transmission line location d_{max}).**
4. **A circle of constant normalized resistance will also intersect this point. Read or interpolate the value of the normalized resistance to determine the *VSWR*.**

Example: Find the *VSWR* for

$$Z_{R1} = 25 + j\,100\ \Omega \ \ ; \ \ Z_{R2} = 25 - j100\Omega \qquad (Z_0 = 50\ \Omega)$$


Circle with constant | Γ |
1
0.5
0.2
0
0.2
0.5
1
2
3
5
-0.2
-0.5
3
2
-1
z_{R1}
Γ_{R1}
Γ_{R2}
z_{R2}
Circle of constant
conductance r = 10.4
$z(d_{max})$=10.4
For both loads
VSWR = 10.4

Given $Z(\mathrm{d}) \Longleftrightarrow$ Find $Y(\mathrm{d})$

Note: The normalized impedance and admittance are defined as

$$z(d) = \frac{1+\Gamma(d)}{1-\Gamma(d)} \qquad y(d) = \frac{1-\Gamma(d)}{1+\Gamma(d)}$$

Since

$$\Gamma\left(d+\frac{\lambda}{4}\right) = -\Gamma(d)$$

$$\Rightarrow \quad z\left(d+\frac{\lambda}{4}\right) = \frac{1+\Gamma\left(d+\frac{\lambda}{4}\right)}{1-\Gamma\left(d+\frac{\lambda}{4}\right)} = \frac{1-\Gamma(d)}{1+\Gamma(d)} = y(d)$$

Keep in mind that the equality

$$z\left(d+\frac{\lambda}{4}\right)=y(d)$$

is only valid for normalized impedance and admittance. The actual values are given by

$$Z\left(d+\frac{\lambda}{4}\right)=Z_0\cdot z\left(d+\frac{\lambda}{4}\right)$$

$$Y(d)=Y_0\cdot y(d)=\frac{y(d)}{Z_0}$$

where $Y_0=1/Z_0$ is the characteristic admittance of the transmission

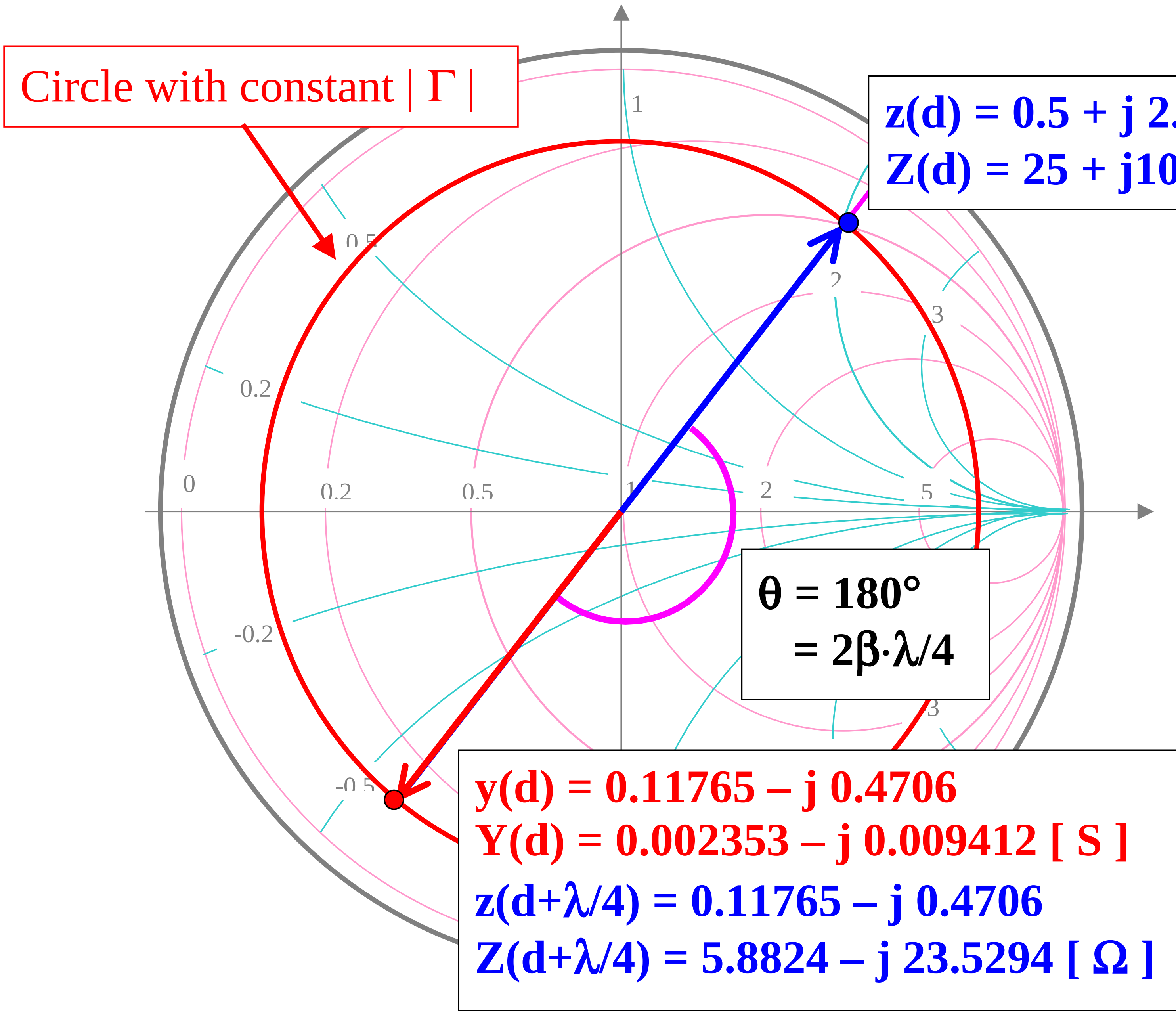
line.
The graphical step-by-step procedure is:

1. **Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.**
2. **Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.**
3. **The normalized admittance is located at a point on the circle of constant $|\Gamma|$ which is diametrically opposite to the normalized impedance.**

Example: Given

$$Z_R = 25 + j\,100\ \Omega \qquad \text{with} \qquad Z_0 = 50\ \Omega$$

find Y_R.


Circle with constant | Γ |
z(d) = 0.5 + j 2.0
Z(d) = 25 + j100 [Ω]
θ = 180°
= 2β·λ/4
y(d) = 0.11765 – j 0.4706
Y(d) = 0.002353 – j 0.009412 [S]
z(d+λ/4) = 0.11765 – j 0.4706
Z(d+λ/4) = 5.8824 – j 23.5294 [Ω]
0
0.2
0.5
1
2
5
3
-0.2
-0.5

The Smith chart can be used for line admittances, by shifting the space reference to the admittance location. After that, one can move on the chart just reading the numerical values as representing admittances.

Let's review the impedance-admittance terminology:

Impedance = Resistance + j Reactance

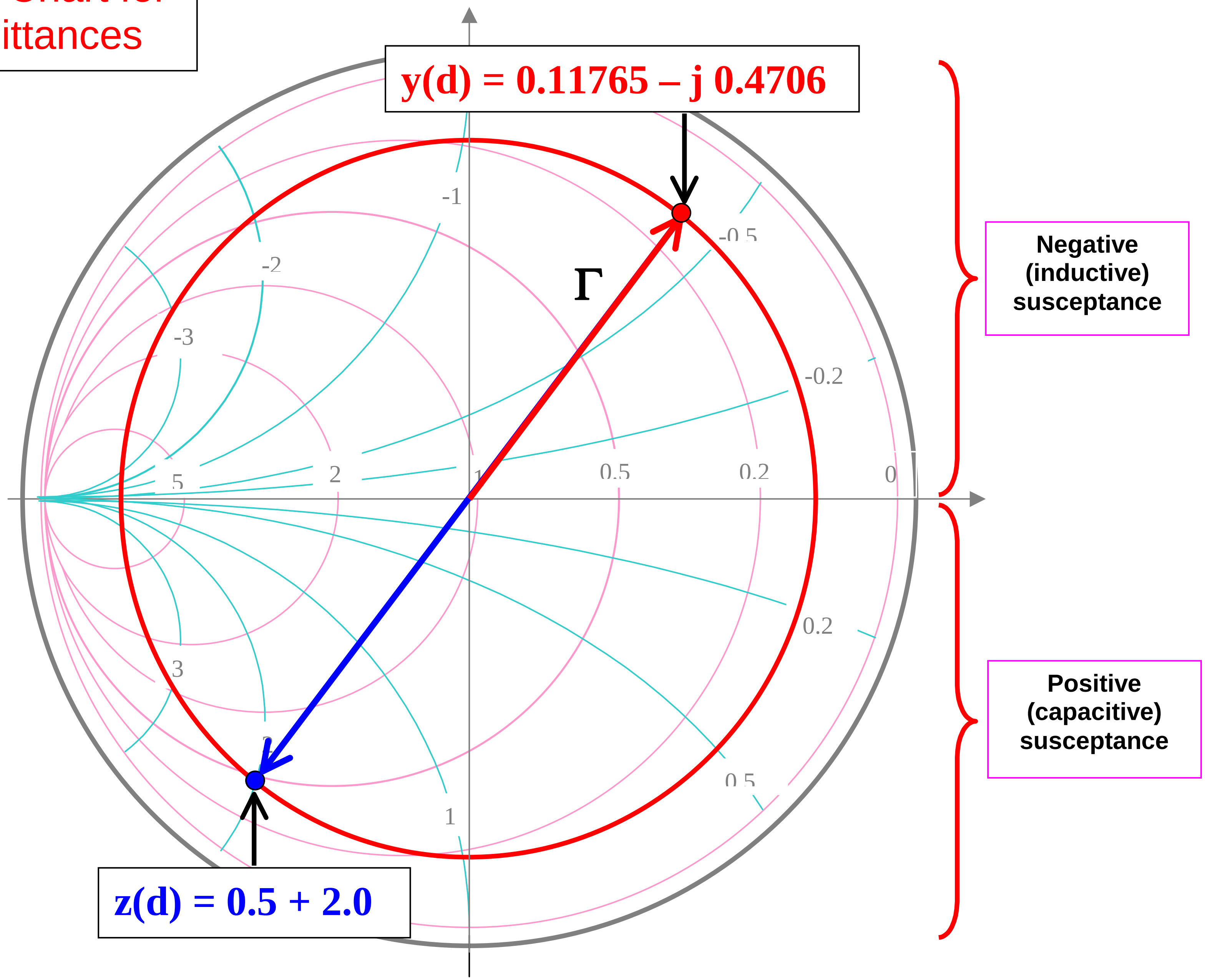
$$Z \quad = \quad R \qquad + \qquad jX$$

Admittance = Conductance + j Susceptance

$$Y \quad = \quad G \qquad + \qquad jB$$

On the impedance chart, the correct reflection coefficient is always represented by the vector corresponding to the normalized impedance. Charts specifically prepared for admittances are modified to give the correct reflection coefficient in correspondence of admittance.

Smith Chart for Admittances



Since related impedance and admittance are on opposite sides of the same Smith chart, the imaginary parts always have different sign.

Therefore, a positive (inductive) reactance corresponds to a negative (inductive) susceptance, while a negative (capacitive) reactance corresponds to a positive (capacitive) susceptance.

Numerically, we have

$$z = r + j\,x \qquad\qquad y = g + j\,b = \frac{1}{r + j\,x}$$

$$y = \frac{r - j\,x}{(r + j\,x)(r - j\,x)} = \frac{r - jx}{r^2 + x^2}$$

$$\Rightarrow \qquad g = \frac{r}{r^2 + x^2} \qquad\qquad b = -\frac{x}{r^2 + x^2}$$