

PRINCIPLES OF COMMUNICATIONS

Systems, Modulation,
and Noise

SEVENTH EDITION

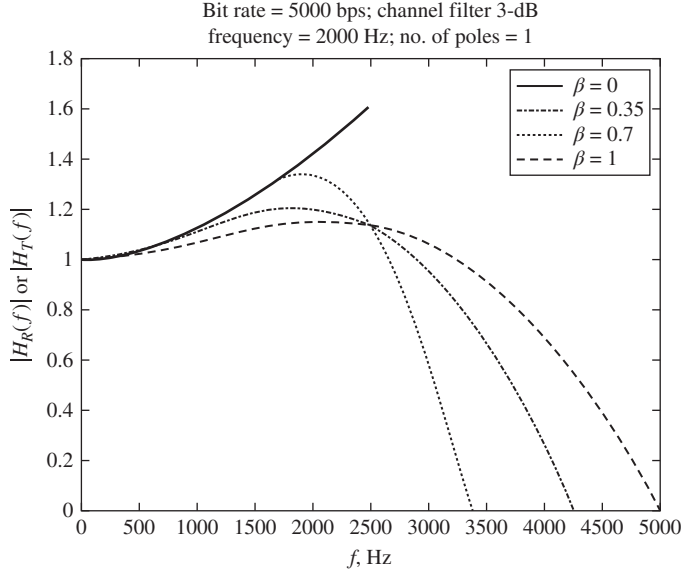
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**Figure 5.10**

Transmitter and receiver filter amplitude responses that implement the zero-ISI condition assuming a first-order Butterworth channel filter and raised-cosine pulse shapes.

Now $|H_C(f)|$ is fixed (the channel is whatever it is) and $P_{RC}(f)$ is specified. Suppose we want the transmitter and receiver filter amplitude responses to be the same. Then, solving (5.46) with $|H_T(f)| = |H_R(f)|$, we have

$$|H_T(f)|^2 = |H_R(f)|^2 = \frac{AP_{RC}(f)}{|H_C(f)|} \quad (5.54)$$

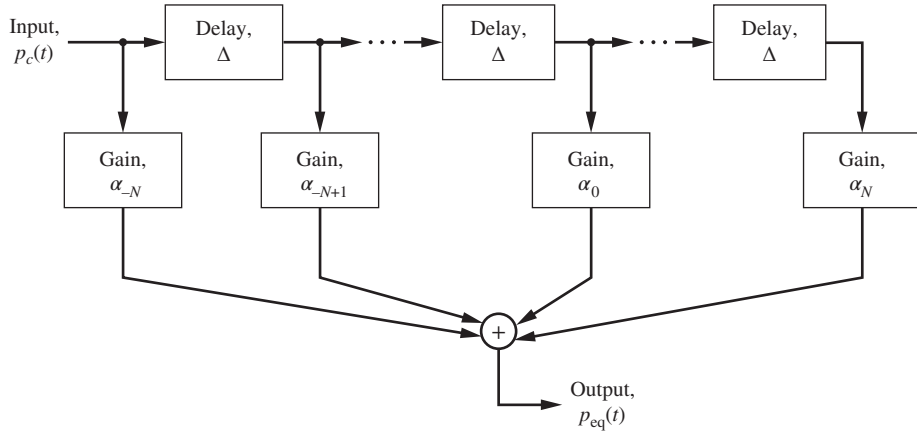
or

$$|H_T(f)| = |H_R(f)| = \frac{A^{1/2}P_{RC}^{1/2}(f)}{|H_C(f)|^{1/2}} \quad (5.55)$$

This amplitude response is shown in Figure 5.10 for raised-cosine spectra of various roll-off factors and for a channel filter assumed to have a first-order Butterworth amplitude response. We have not accounted for the effects of additive noise. If the noise spectrum is flat, the only change would be another multiplicative constant. The constants are arbitrary since they multiply both signal and noise alike.

■ 5.5 ZERO-FORCING EQUALIZATION

In the previous section, it was shown how to choose transmitter and receiver filter amplitude responses, given a certain channel filter, to provide output pulses satisfying the zero-ISI condition. In this section, we present a procedure for designing a filter that will accept a channel output pulse response not satisfying the zero-ISI condition and produce a pulse at its output that has N zero-valued samples on either side of its maximum sample value taken to

**Figure 5.11**

A transversal filter implementation for equalization of intersymbol interference.

be 1 for convenience. This filter will be called a zero-forcing equalizer. We specialize our considerations of an equalization filter to a particular form—a transversal or tapped-delay-line filter. Figure 5.11 shows the block diagram of such a filter.

There are at least two reasons for considering a transversal structure for the purpose of equalization. First, it is simple to analyze. Second, it is easy to mechanize by electronic means (i.e., transmission line delays and analog multipliers) at high frequencies and by digital signal processors at lower frequencies.

Let the pulse response of the channel output be $p_c(t)$. The output of the equalizer in response to $p_c(t)$ is

$$p_{eq}(t) = \sum_{n=-N}^N \alpha_n p_c(t - n\Delta) \quad (5.56)$$

where Δ is the tap spacing and the total number of transversal filter taps is $2N + 1$. We want $p_{eq}(t)$ to satisfy Nyquist's pulse-shaping criterion, which we will call the *zero-ISI condition*. Since the output of the equalizer is sampled every T seconds, it is reasonable that the tap spacing be $\Delta = T$. The zero-ISI condition therefore becomes

$$\begin{aligned} p_{eq}(mT) &= \sum_{n=-N}^N \alpha_n p_c[(m - n)T] \\ &= \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases} \quad m = 0, \pm 1, \pm 2, \dots, \pm N \end{aligned} \quad (5.57)$$

Note that the zero-ISI condition can be satisfied at only $2N$ time instants because there are only $2N + 1$ coefficients to be selected in (5.57) and the output of the filter for $t = 0$ is forced

to be 1. Defining the matrices (actually column matrices or vectors for the first two)

$$[P_{\text{eq}}] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \left. \begin{array}{l} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{array} \right\} \begin{array}{l} N \text{ zeros} \\ \\ \\ N \text{ zeros} \end{array} \quad (5.58)$$

$$[A] = \begin{bmatrix} \alpha_{-N} \\ \alpha_{-N+1} \\ \vdots \\ \alpha_N \end{bmatrix} \quad (5.59)$$

and

$$[P_c] = \begin{bmatrix} p_c(0) & p_c(-T) & \cdots & p_c(-2NT) \\ p_c(T) & p_c(0) & \cdots & p_c[(-2N+1)T] \\ \vdots & & & \vdots \\ p_c(2NT) & & & p_c(0) \end{bmatrix} \quad (5.60)$$

it follows that (5.57) can be written as the matrix equation

$$[P_{\text{eq}}] = [P_c][A] \quad (5.61)$$

The method of solution of the zero-forcing coefficients is now clear. Since $[P_{\text{eq}}]$ is specified by the zero-ISI condition, all we must do is multiply through by the inverse of $[P_c]$. The desired coefficient matrix $[A]$ is then the middle column of $[P_c]^{-1}$, which follows by multiplying $[P_c]^{-1}$ times $[P_{\text{eq}}]$:

$$[A] = [P_c]^{-1}[P_{\text{eq}}] = [P_c]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \text{middle column of } [P_c]^{-1} \quad (5.62)$$

EXAMPLE 5.7

Consider a channel for which the following sample values of the channel pulse response are obtained:

$$\begin{aligned} p_c(-3T) &= 0.02 & p_c(-2T) &= -0.05 & p_c(-T) &= 0.2 & p_c(0) &= 1.0 \\ p_c(T) &= 0.3 & p_c(2T) &= -0.07 & p_c(3T) &= 0.03 \end{aligned}$$

The matrix $[P_c]$ is

$$[P_c] = \begin{bmatrix} 1.0 & 0.2 & -0.05 \\ 0.3 & 1.0 & 0.2 \\ -0.07 & 0.3 & 1.0 \end{bmatrix} \quad (5.63)$$

and the inverse of this matrix is

$$[P_c]^{-1} = \begin{bmatrix} 1.0815 & -0.2474 & 0.1035 \\ -0.3613 & 1.1465 & -0.2474 \\ 0.1841 & -0.3613 & 1.0815 \end{bmatrix} \quad (5.64)$$

Thus, by (5.62)

$$[A] = \begin{bmatrix} 1.0815 & -0.2474 & 0.1035 \\ -0.3613 & 1.1465 & -0.2474 \\ 0.1841 & -0.3613 & 1.0815 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.2474 \\ 1.1465 \\ -0.3613 \end{bmatrix} \quad (5.65)$$

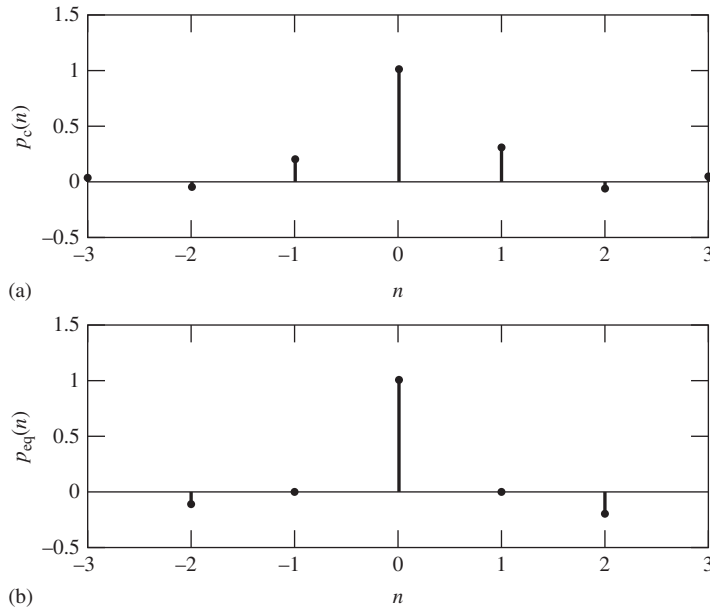


Figure 5.12

Samples for (a) an assumed channel response and (b) the output of a zero-forcing equalizer of length 3.

Using these coefficients, the equalizer output is

$$p_{\text{eq}}(m) = -0.2474p_c[(m+1)T] + 1.1465p_c(mT) - 0.3613p_c[(m-1)T], \quad m = \dots, -1, 0, 1, \dots \quad (5.66)$$

Putting values in shows that $p_{\text{eq}}(0) = 1$ and that the single samples on either side of $p_{\text{eq}}(0)$ are zero. Samples more than one away from the center sample are not necessarily zero for this example. Calculation using the extra samples for $p_c(nT)$ gives $p_c(-2T) = -0.1140$ and $p_c(2T) = -0.1961$. Samples for the channel and the equalizer outputs are shown in Figure 5.12. ■

5.6 EYE DIAGRAMS

We now consider eye diagrams that, although not a quantitative measure of system performance, are simple to construct and give significant insight into system performance. An eye diagram is constructed by plotting overlapping k -symbol segments of a baseband signal. In other words, an eye diagram can be displayed on an oscilloscope by triggering the time sweep of the oscilloscope, as shown in Figure 5.13, at times $t = nkT_s$ where T_s is the symbol period, kT_s is the eye period, and n is an integer. A simple example will demonstrate the process of generating an eye diagram.

EXAMPLE 5.8

Consider the eye diagram of a bandlimited digital NRZ baseband signal. In this example the signal is generated by passing an NRZ waveform through a third-order Butterworth filter as illustrated in Figure 5.13. The filter bandwidth is normalized to the symbol rate. In other words, if the symbol rate of the NRZ waveform is 1000 symbols per second, and the normalized filter bandwidth is $B_N = 0.6$, the filter bandwidth is 600 hertz. The eye diagrams corresponding to the signal at the filter output are those illustrated in Figure 5.14 for normalized bandwidths, B_N , of 0.4, 0.6, 1.0, and 2.0. Each of the four eye diagrams span $k = 4$ symbols. Sampling is performed at 20 samples/symbol and therefore the sampling index ranges from 1 to 80 as shown. The effect of bandlimiting by the filter, leading to intersymbol interference, on the eye diagram is clearly seen.

We now look at an eye diagram in more detail. Figure 5.15 shows the top pane of Figure 5.14 ($B_N = 0.4$), in which two symbols are illustrated rather than four. Observation of Figure 5.15 suggests that the eye diagram is composed of two fundamental waveforms, each of which approximates a sinewave. One waveform goes through two periods in the two symbol eyes and the other waveform goes through a single period. A little thought shows that the high-frequency waveform corresponds to the binary sequences 01 or 10 while the low-frequency waveform corresponds to the binary sequences 00 or 11.

Also shown in Figure 5.15 is the optimal sampling time, which is when the eye is most open. Note that for significant bandlimiting the eye will be more closed due to intersymbol interference. This shrinkage of the eye opening due to ISI is labeled amplitude jitter, A_j . Referring back to Figure 5.14 we see that increasing the filter bandwidth decreases the amplitude jitter. When we consider the effects of

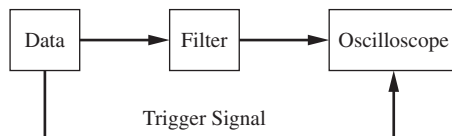


Figure 5.13

Simple technique for generating an eye diagram for a bandlimited signal.

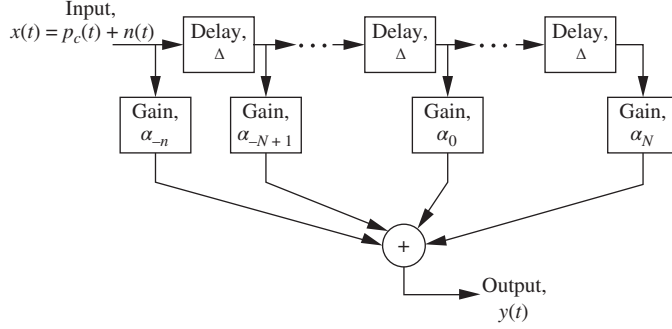


Figure 9.31
Transversal filter
implementation for
equalization of intersymbol
interference.

9.9 EQUALIZATION

As explained in Section 9.7, an equalization filter can be used to combat channel-induced distortion caused by perturbations such as multipath propagation or bandlimiting due to filters. According to (9.186), a simple approach to the idea of equalization leads to the concept of an inverse filter. As in Chapter 5, we specialize our considerations of an equalization filter to a particular form—a transversal or tapped-delay-line filter the block diagram of which is repeated in Figure 9.31.²⁰

We can take two approaches to determining the tap weights, $\alpha_{-N}, \dots, \alpha_0, \dots, \alpha_N$ in Figure 9.31 for given channel conditions. One is *zero-forcing*, and the other is *minimization of mean-square error*. We briefly review the first method, including a consideration of noise effects, and then consider the second.

9.9.1 Equalization by Zero-Forcing

In Chapter 5 it was shown how the pulse response of the channel output, $p_c(t)$, could be forced to have a maximum value of 1 at the desired sampling time with N samples of 0 on either side of the maximum by properly choosing the tap weights of a $(2N + 1)$ -tap transversal filter. For a desired equalizer output at the sampling times of

$$\begin{aligned}
 p_{\text{eq}}(mT) &= \sum_{n=-N}^N \alpha_n p_c[(m-n)T] \\
 &= \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases} \quad m = 0, \pm 1, \pm 2, \dots, \pm N
 \end{aligned} \tag{9.209}$$

the solution was to find the middle column of the inverse of the channel response matrix $[P_c]$:

$$[P_{\text{eq}}] = [P_c][A] \tag{9.210}$$

²⁰For an excellent overview of equalization, see S. Quereschi, “Adaptive Equalization,” *Proc. of the IEEE*, **73**, 1349–1387, September 1985.

where the various matrices are defined as

$$[P_{\text{eq}}] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} N \text{ zeros} \\ \\ \\ \\ \\ N \text{ zeros} \end{array} \quad (9.211)$$

$$[A] = \begin{bmatrix} \alpha_{-N} \\ \alpha_{-N+1} \\ \vdots \\ \alpha_N \end{bmatrix} \quad (9.212)$$

and

$$[P_c] = \begin{bmatrix} p_c(0) & p_c(-T) & \cdots & p_c(-2NT) \\ p_c(T) & p_c(0) & \cdots & p_c(-2N+1)T \\ \vdots & & & \vdots \\ p_c(2NT) & & & p_c(0) \end{bmatrix} \quad (9.213)$$

That is, the equalizer coefficient matrix is given by

$$[A]_{\text{opt}} = [P_c]^{-1} [P_{\text{eq}}] = \text{middle column of } [P_c]^{-1} \quad (9.214)$$

The equalizer response for delays less than $-NT$ or greater than NT are not necessarily zero. Since the zero-forcing equalization procedure only takes into account the received pulse sample values while ignoring the noise, it is not surprising that its noise performance may be poor in some channels. In fact, in some cases, the noise spectrum is enhanced considerably at certain frequencies by a zero-forcing equalizer as a plot of its frequency response reveals:

$$H_{\text{eq}}(f) = \sum_{n=-N}^N \alpha_n \exp(-j2\pi n f T) \quad (9.215)$$

To assess the effects of noise, consider the input-output relation for the transversal filter with a signal pulse plus Gaussian noise of power spectral density $G_n(f) = \frac{N_0}{2} \Pi\left(\frac{f}{2B}\right)$ at its

input. The output can be written as

$$\begin{aligned}
 y(mT) &= \sum_{l=-N}^N \alpha_l \{p_c[(m-l)T] + n[(m-l)T]\} \\
 &= \sum_{l=-N}^N \alpha_l p_c[(m-l)T] + \sum_{l=-N}^N \alpha_l n[(m-l)T] \\
 &= p_{\text{eq}}(mT) + N_m, \quad m = \dots, -2, -1, 0, 1, 2, \dots
 \end{aligned} \tag{9.216}$$

The random variables $\{N_m\}$ are zero-mean, Gaussian, and have variance

$$\begin{aligned}
 \sigma_N^2 &= E\{N_k^2\} \\
 &= E\left\{\sum_{j=-N}^N \alpha_j n[(k-j)T] \sum_{l=-N}^N \alpha_l n[(k-l)T]\right\} \\
 &= E\left\{\sum_{j=-N}^N \sum_{l=-N}^N \alpha_j \alpha_l n[(k-j)T] n[(k-l)T]\right\} \\
 &= \sum_{j=-N}^N \sum_{l=-N}^N \alpha_j \alpha_l E\{n[(k-j)T] n[(k-l)T]\} \\
 &= \sum_{j=-N}^N \sum_{l=-N}^N \alpha_j \alpha_l R_n[(j-l)T]
 \end{aligned} \tag{9.217}$$

where

$$R_n(\tau) = \mathfrak{F}^{-1}[G_n(f)] = \mathfrak{F}^{-1}\left[\frac{N_0}{2}\Pi\left(\frac{f}{2B}\right)\right] = N_0 B \text{sinc}(2B\tau) \tag{9.218}$$

If it is assumed that $2BT = 1$ (consistent with the sampling theorem), then

$$R_n[(j-l)T] = N_0 B \text{sinc}(j-l) = \frac{N_0}{2T} \text{sinc}(j-l) = \begin{cases} \frac{N_0}{2T}, & j=l \\ 0, & j \neq l \end{cases} \tag{9.219}$$

and (9.217) becomes

$$\sigma_N^2 = \frac{N_0}{2T} \sum_{j=-N}^N \alpha_j^2 \tag{9.220}$$

For a sufficiently long equalizer, the signal component of the output, assuming binary transmission, can be taken as ± 1 equally likely. The probability of error is then

$$\begin{aligned}
 P_E &= \frac{1}{2} \Pr(-1 + N_m > 0) + \frac{1}{2} \Pr(1 + N_m < 0) \\
 &= \Pr(N_m > 1) = \Pr(N_m < -1) \quad (\text{by symmetry of the noise pdf})
 \end{aligned}$$

$$\begin{aligned}
&= \int_1^\infty \frac{\exp(-\eta^2 / (2\sigma_N^2))}{\sqrt{2\pi\sigma_N^2}} d\eta = Q\left(\frac{1}{\sigma_N}\right) \\
&= Q\left(\frac{1}{\sqrt{\frac{N_0}{2T} \sum_j \alpha_j^2}}\right) = Q\left(\sqrt{\frac{2 \times 1^2 \times T}{N_0 \sum_j \alpha_j^2}}\right) = Q\left(\sqrt{\frac{1}{\sum_j \alpha_j^2} \frac{2E_b}{N_0}}\right) \quad (9.221)
\end{aligned}$$

From (9.221) it is seen that performance is degraded in proportion to $\sum_{j=-N}^N \alpha_j^2$, which is a factor that directly enhances the output noise.

EXAMPLE 9.11

Consider the following pulse samples at a channel output:

$$\{p_c(n)\} = \{-0.01 \ 0.05 \ 0.004 \ -0.1 \ 0.2 \ -0.5 \ 1.0 \ 0.3 \ -0.4 \ 0.04 \ -0.02 \ 0.01 \ 0.001\}$$

Obtain the five-tap zero-forcing equalizer coefficients and plot the magnitude of the equalizer's frequency response. By what factor is the signal-to-noise ratio worsened due to noise enhancement?

Solution

The matrix $[P_c]$, from (9.213), is

$$[P_c] = \begin{bmatrix} 1 & -0.5 & 0.2 & -0.1 & 0.004 \\ 0.3 & 1 & -0.5 & 0.2 & -0.1 \\ -0.4 & 0.3 & 1 & -0.5 & 0.2 \\ 0.04 & -0.4 & 0.3 & 1 & -0.5 \\ -0.02 & 0.04 & -0.4 & 0.3 & 1 \end{bmatrix} \quad (9.222)$$

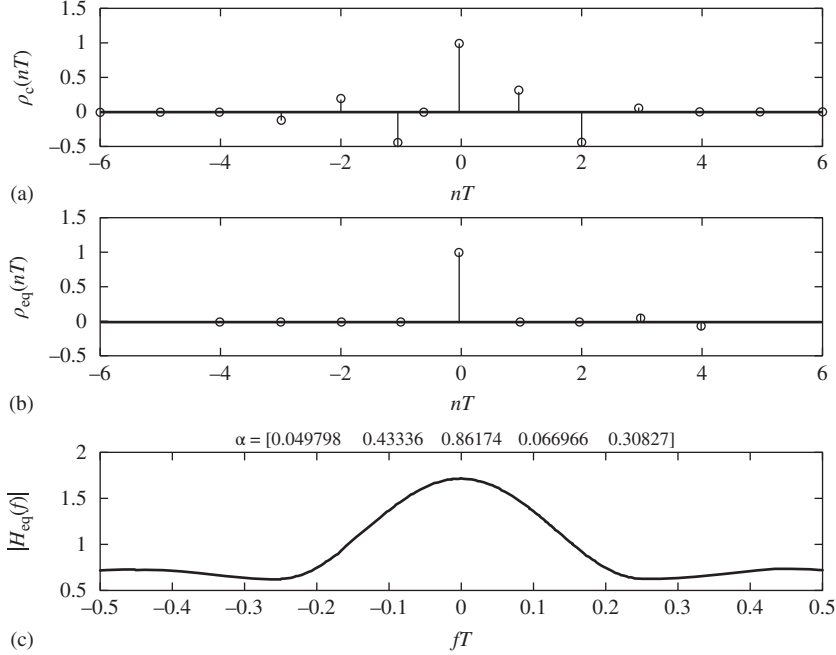
The equalizer coefficients are the middle column of $[P_c]^{-1}$, which is

$$[P_c]^{-1} = \begin{bmatrix} 0.889 & 0.435 & 0.050 & 0.016 & 0.038 \\ -0.081 & 0.843 & 0.433 & 0.035 & 0.016 \\ 0.308 & 0.067 & 0.862 & 0.433 & 0.050 \\ -0.077 & 0.261 & 0.067 & 0.843 & 0.435 \\ 0.167 & -0.077 & 0.308 & -0.081 & 0.890 \end{bmatrix} \quad (9.223)$$

Therefore, the coefficient vector is

$$[A]_{\text{opt}} = [P_c]^{-1} [P_{\text{eq}}] = \begin{bmatrix} 0.050 \\ 0.433 \\ 0.862 \\ 0.067 \\ 0.308 \end{bmatrix} \quad (9.224)$$

Plots of the input and output sequences are given in Figure 9.32(a) and (b), respectively, and a plot of the equalizer frequency response magnitude is shown in Figure 9.32(c). There is considerable enhancement of the output noise spectrum at low frequencies as is evident from the frequency response.

**Figure 9.32**

(a) Input and (b) output sample sequences for a five-tap zero-forcing equalizer. (c) Equalizer frequency response.

Depending on the received pulse shape, the noise enhancement may be at higher frequencies in other cases. The noise enhancement, or degradation, factor in this example is

$$\sum_{j=-4}^4 \alpha_j^2 = 1.0324 = 0.14 \text{ dB} \quad (9.225)$$

which is not severe in this case. ■

9.9.2 Equalization by MMSE

Suppose that the desired output from the transversal filter equalizer of Figure 9.31 is $d(t)$. A minimum mean-squared error (MMSE) criterion then seeks the tap weights that minimize the mean-squared error between the desired output from the equalizer and its actual output. Since this output includes noise, we denote it by $z(t)$ to distinguish it from the pulse response of the equalizer. The MMSE criterion is therefore expressed as

$$\mathcal{E} = E \{ [z(t) - d(t)]^2 \} = \text{minimum} \quad (9.226)$$

where, if $y(t)$ is the equalizer input including noise, the equalizer output is

$$z(t) = \sum_{n=-N}^N \alpha_n y(t - n\Delta) \quad (9.227)$$

Since $\mathcal{E}\{\cdot\}$ is a concave (bowl-shaped) function of the tap weights, a set of sufficient conditions for minimizing the tap weights is

$$\frac{\partial \mathcal{E}}{\partial \alpha_m} = 0 = 2E \left\{ [z(t) - d(t)] \frac{\partial z(t)}{\partial \alpha_m} \right\}, \quad m = 0, \pm 1, \dots, \pm N \quad (9.228)$$

Substituting (9.227) into (9.228) and carrying out the differentiation, we obtain the conditions

$$E \{ [z(t) - d(t)] y(t - m\Delta) \} = 0, \quad m = 0, \pm 1, \pm 2, \dots, \pm N \quad (9.229)$$

or

$$R_{yz}(m\Delta) = R_{yd}(m\Delta) = 0, \quad m = 0, \pm 1, \pm 2, \dots, \pm N \quad (9.230)$$

where

$$R_{yz}(\tau) = E[y(t)z(t + \tau)] \quad (9.231)$$

and

$$R_{yd}(\tau) = E[y(t)d(t + \tau)] \quad (9.232)$$

are the cross-correlations of the received signal with the equalizer output and with the data, respectively.

Using the expression (9.227) for $z(t)$ in (9.230), these conditions can be expressed as the matrix equation²¹

$$[R_{yy}][A]_{\text{opt}} = [R_{yd}] \quad (9.233)$$

where

$$[R_{yy}] = \begin{bmatrix} R_{yy}(0) & R_{yy}(\Delta) & \cdots & R_{yy}(2N\Delta) \\ R_{yy}(-\Delta) & R_{yy}(0) & \cdots & R_{yy}[2(N-1)\Delta] \\ \vdots & & & \vdots \\ R_{yy}(-2N\Delta) & & \cdots & R_{yy}(0) \end{bmatrix} \quad (9.234)$$

and

$$[R_{yd}] = \begin{bmatrix} R_{yd}(-N\Delta) \\ R_{yd}[-(N-1)\Delta] \\ \vdots \\ R_{yd}(N\Delta) \end{bmatrix} \quad (9.235)$$

and $[A]$ is defined by (9.212). Note that these conditions for the optimum tap weights using the MMSE criterion are similar to the conditions for the zero-forcing weights, except correlation-function samples are used instead of pulse-response samples.

The solution to (9.233) is

$$[A]_{\text{opt}} = [R_{yy}]^{-1}[R_{yd}] \quad (9.236)$$

²¹These are known as the Wiener-Hopf equations. See S. Haykin, *Adaptive Filter Theory*, 3rd ed., Upper Saddle River, NJ: Prentice Hall, 1996.

which requires knowledge of the correlation matrices. The mean-squared error is

$$\begin{aligned}
 \mathcal{E} &= E \left\{ \left[\sum_{n=-N}^N \alpha_n y(t - n\Delta) - d(t) \right]^2 \right\} \\
 &= E \left\{ d^2(t) - 2d(t) \sum_{n=-N}^N \alpha_n y(t - n\Delta) + \sum_{m=-N}^N \sum_{n=-N}^N \alpha_m \alpha_n y(t - m\Delta) y(t - n\Delta) \right\} \\
 &= E \{ d^2(t) \} - 2 \sum_{n=-N}^N \alpha_n E \{ d(t) y(t - n\Delta) \} \\
 &\quad + \sum_{m=-N}^N \sum_{n=-N}^N \alpha_m \alpha_n E \{ y(t - m\Delta) y(t - n\Delta) \} \\
 &= \sigma_d^2 - 2 \sum_{n=-N}^N \alpha_n R_{yd}(n\Delta) + \sum_{m=-N}^N \sum_{n=-N}^N \alpha_m \alpha_n R_{yy}[(m - n)\Delta] \\
 &= \sigma_d^2 - 2 [A]^T [R_{yd}] + [A]^T [R_{yy}] [A] \tag{9.237}
 \end{aligned}$$

where the superscript T denotes the matrix transpose and $\sigma_d^2 = E [d^2(t)]$. For the optimum weights, (9.236), this becomes

$$\begin{aligned}
 \mathcal{E}_{\min} &= \sigma_d^2 - 2 \{ [R_{yy}]^{-1} [R_{yd}] \}^T [R_{yd}] + \{ [R_{yy}]^{-1} [R_{yd}] \}^T [R_{yy}] \{ [R_{yy}]^{-1} [R_{yd}] \} \\
 &= \sigma_d^2 - 2 \{ [R_{yd}]^T [R_{yy}]^{-1} \} [R_{yd}] + [R_{yd}]^T [R_{yy}]^{-1} [R_{yy}] \{ [R_{yy}]^{-1} [R_{yd}] \} \\
 &= \sigma_d^2 - 2 [R_{yd}]^T [A]_{\text{opt}} + [R_{yd}]^T [A]_{\text{opt}} \\
 &= \sigma_d^2 - [R_{yd}]^T [A]_{\text{opt}} \tag{9.238}
 \end{aligned}$$

where the matrix relation $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$ has been used along with the fact that the autocorrelation matrix is symmetric.

The question remains as to the choice for the time delay Δ between adjacent taps. If the channel distortion is due to multiple transmission paths (multipath) with the delay of a strong component equal to a fraction of a bit period, then it may be advantageous to set Δ equal to that expected fraction of a bit period (called a *fractionally spaced equalizer*).²² On the other hand, if the shortest multipath delay is several bit periods, then it would make sense to set $\Delta = T$.

²²See J. R. Treichler, I. Fijalkow, and C. R. Johnson, Jr., "Fractionally Spaced Equalizers," *IEEE Signal Proc. Mag.*, 65–81, May 1996.

EXAMPLE 9.12

Consider a channel consisting of a direct path and a single indirect path plus additive Gaussian noise. Thus, the channel output is

$$y(t) = A_0 d(t) + \beta A_0 d(t - \tau_m) + n(t) \quad (9.239)$$

where it is assumed that carrier demodulation has taken place so $d(t) = \pm 1$ in T -second bit periods is the data with assumed autocorrelation function $R_{dd}(\tau) = \Lambda(\tau/T)$ (i.e., a random coin-toss sequence); A_0 is the signal amplitude. The strength of the multipath component relative to the direct component is β and its relative delay is τ_m . The noise $n(t)$ is assumed to be bandlimited with power spectral density $S_n(f) = \frac{N_0}{2} \Pi\left(\frac{f}{2B}\right)$ W/Hz so that its autocorrelation function is $R_{nn}(\tau) = N_0 B \text{sinc}(2B\tau)$ where it is assumed that $2BT = 1$. Find the coefficients of an MMSE three-tap equalizer with tap spacing $\Delta = T$ assuming that $\tau_m = T$.

Solution

The autocorrelation function of $y(t)$ is

$$\begin{aligned} R_{yy}(\tau) &= E\{y(t)y(t+\tau)\} \\ &= E\{[A_0 d(t) + \beta A_0 d(t - \tau_m) + n(t)][A_0 d(t+\tau) + \beta A_0 d(t+\tau - \tau_m) + n(t+\tau)]\} \\ &= (1 + \beta^2) A_0^2 R_{dd}(\tau) + R_{nn}(\tau) + \beta A_0^2 [R_{dd}(\tau - T) + R_{dd}(\tau + T)] \end{aligned} \quad (9.240)$$

In a similar fashion, we find

$$\begin{aligned} R_{yd}(\tau) &= E[y(t)d(t+\tau)] \\ &= A_0 R_{dd}(\tau) + \beta A_0 R_{dd}(\tau + T) \end{aligned} \quad (9.241)$$

Using (9.234) with $N = 3$, $\Delta = T$, and $2BT = 1$ we find

$$[R_{yy}] = \begin{bmatrix} (1 + \beta^2) A_0^2 + N_0 B & \beta A_0^2 & 0 \\ \beta A_0^2 & (1 + \beta^2) A_0^2 + N_0 B & \beta A_0^2 \\ 0 & \beta A_0^2 & (1 + \beta^2) A_0^2 + N_0 B \end{bmatrix} \quad (9.242)$$

and

$$[R_{yd}] = \begin{bmatrix} R_{yd}(-T) \\ R_{yd}(0) \\ R_{yd}(T) \end{bmatrix} = \begin{bmatrix} \beta A_0 \\ A_0 \\ 0 \end{bmatrix} \quad (9.243)$$

The condition (9.233) for the optimum weights becomes

$$\begin{bmatrix} (1 + \beta^2) A_0^2 + N_0 B & \beta A_0^2 & 0 \\ \beta A_0^2 & (1 + \beta^2) A_0^2 + N_0 B & \beta A_0^2 \\ 0 & \beta A_0^2 & (1 + \beta^2) A_0^2 + N_0 B \end{bmatrix} \begin{bmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \beta A_0 \\ A_0 \\ 0 \end{bmatrix} \quad (9.244)$$

We may make these equations dimensionless by factoring out $N_0 B$ (recall that $2BT = 1$ by assumption) and defining new weights $c_i = A_0 \alpha_i$, which gives

$$\begin{bmatrix} (1 + \beta^2) \frac{2E_b}{N_0} + 1 & 2\beta \frac{E_b}{N_0} & 0 \\ 2\beta \frac{E_b}{N_0} & (1 + \beta^2) \frac{2E_b}{N_0} + 1 & 2\beta \frac{E_b}{N_0} \\ 0 & 2\beta \frac{E_b}{N_0} & (1 + \beta^2) \frac{2E_b}{N_0} + 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2\beta \frac{E_b}{N_0} \\ 2 \frac{E_b}{N_0} \\ 0 \end{bmatrix} \quad (9.245)$$

Table 9.6 Bit Error Performance of an MMSE Equalizer in Multipath

E_b/N_0 , dB	No. of bits	P_b , no equal.	P_b , equal.	P_b , Gauss noise only
10	200,000	5.7×10^{-3}	4.4×10^{-4}	3.9×10^{-6}
11	300,000	2.7×10^{-3}	1.4×10^{-4}	2.6×10^{-7}
12	300,000	1.2×10^{-3}	4.3×10^{-5}	9.0×10^{-9}

where $\frac{E_b}{N_0} \triangleq \frac{A_0^2 T}{N_0}$. For numerical values, we assume that $\frac{E_b}{N_0} = 10$ and $\beta = 0.5$, which gives

$$\begin{bmatrix} 26 & 10 & 0 \\ 10 & 26 & 10 \\ 0 & 10 & 26 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix} \quad (9.246)$$

or, finding the inverse of the modified R_{yy} matrix using MATLAB, we get

$$\begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0.0465 & -0.0210 & 0.0081 \\ -0.0210 & 0.0546 & -0.0210 \\ 0.0081 & -0.0210 & 0.0465 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix} \quad (9.247)$$

giving finally that

$$\begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0.045 \\ 0.882 \\ -0.339 \end{bmatrix} \quad (9.248)$$

We can find the minimum mean-square error according to (9.238). We need the optimum weights, given by (9.248), and the cross-correlation matrix, given by (9.243). Assuming that $A_0 = 1$, it follows that $\alpha_j = \frac{1}{A_0} c_j = c_j$. Also, $\sigma_d^2 = 1$ assuming that $d(t) = \pm 1$. Hence, the minimum mean-square error is

$$\begin{aligned} \epsilon_{\min} &= \sigma_d^2 - [R_{yd}]^T [A]_{\text{opt}} \\ &= 1 - \begin{bmatrix} \beta A_0 & A_0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{bmatrix} \\ &= 1 - \begin{bmatrix} 0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.045 \\ 0.882 \\ -0.339 \end{bmatrix} = 0.095 \end{aligned} \quad (9.249)$$

Evaluation of the equalizer performance in terms of bit error probability requires simulation. Some results are given in Table 9.6, where it is seen that the equalizer provides significant improvement in this case. ■

9.9.3 Tap Weight Adjustment

Two questions remain with regard to setting the tap weights. The first is what should be used for the desired response $d(t)$? In the case of digital signaling, one has two choices.

1. A known data sequence can be sent periodically and used for tap weight adjustment.
2. The detected data can be used if the modem performance is moderately good, since an error probability of only 10^{-2} , for example, still implies that $d(t)$ is correct for 99 out of 100 bits. Algorithms using the detected data as $d(t)$, the desired output, are called *decision-directed*. Often, the equalizer tap weights will be initially adjusted using a known sequence and after settling into nearly optimum operation, the adjustment algorithm will be switched over to a decision-directed mode.

The second question is what procedure should be followed if the sample values of the pulse needed in the zero-forcing criterion or the samples of the correlation function required for the MMSE criterion are not available. Useful strategies to follow in such cases fall under the heading of *adaptive equalization*.

To see how one might implement such a procedure, we note that the mean-squared error (9.237) is a quadratic function of the tap weights with minimum value given by (9.238) for the optimum weights. Thus, the method of steepest descent may be applied. In this procedure, initial values for the weights, $[A]^{(0)}$, are chosen and subsequent values are calculated according to²³

$$[A]^{(k+1)} = [A]^{(k)} + \frac{1}{2}\mu [-\nabla \mathcal{E}^{(k)}], \quad k = 0, 1, 2, \dots \quad (9.250)$$

where the superscript k denotes the k th calculation time and $\nabla \mathcal{E}$ is the gradient, or “slope,” of the error surface. The idea is that starting with an initial guess of the weight vector, then the next closest guess is in the direction of the negative gradient. Clearly, the parameter $\mu/2$ is important in this stepwise approach to the minimum of $\mathcal{E}$, for one of two adverse things can happen: (1) A very small choice for μ means very slow convergence to the minimum of $\mathcal{E}$; (2) Too large of a choice for μ can mean overshoot of the minimum for $\mathcal{E}$ with the result being damped oscillation about the minimum or even divergence from it.²⁴ To guarantee convergence, the adjustment parameter μ should obey the relation

$$0 < \mu < 2/\lambda_{\max} \quad (9.251)$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $[R_{yy}]$ according to Haykin. Another rule of thumb for choosing μ is²⁵

$$0 < \mu < 1 / [(L + 1) \times (\text{signal power})] \quad (9.252)$$

where $L = 2N + 1$. This avoids computing the autocorrelation matrix (a difficult problem with limited data).

Note that use of the steepest descent algorithm does not remove two disadvantages of the optimum weight computation: (1) It is dependent on knowing the correlation matrices $[R_{yd}]$ and $[R_{yy}]$; (2) It is computationally intensive in that matrix multiplications are still necessary (although no matrix inversions), for the gradient of $\mathcal{E}$ can be shown to be

$$\begin{aligned} \nabla \mathcal{E} &= \nabla \{ \sigma_d^2 - 2[A]^T [R_{yd}] + [A]^T [R_{yy}] [A] \} \\ &= -2 [R_{yd}] + 2 [R_{yy}] [A] \end{aligned} \quad (9.253)$$

²³See Haykin, 1996, Section 8.2, for a full development.

²⁴Two sources of error in tap weight adjustment are due to the input noise itself and the adjustment noise of the tap weight adjustment algorithm.

²⁵Widrow and Stearns, 1985.

which must be recalculated for each new estimate of the weights. Substituting (9.253) into (9.250) gives

$$[A]^{(k+1)} = [A]^{(k)} + \mu [[R_{yd}] - [R_{yy}] [A]^{(k)}], \quad k = 0, 1, 2, \dots \quad (9.254)$$

An alternative approach, known as the least-mean-square (LMS) algorithm, that avoids both of these disadvantages, replaces the matrices $[R_{yd}]$ and $[R_{yy}]$ with instantaneous data-based estimates. An initial guess for α_m is corrected from step k to step $k + 1$ according to the recursive relationship

$$\alpha_m^{(k+1)} = \alpha_m^{(k)} - \mu y[(k-m)\Delta] \epsilon(k\Delta), \quad m = 0, \pm 1, \dots, \pm N \quad (9.255)$$

where the error is $\epsilon(k\Delta) = y_{\text{eq}}(k\Delta) - d(k\Delta)$ with $y_{\text{eq}}(k\Delta)$ being the equalizer output and $d(k\Delta)$ being a data sequence (either a training sequence or detected data if the tap weights have been adapted sufficiently). Note that some delays may be necessary in aligning the detected data with the equalizer output.

EXAMPLE 9.13

The maximum eigenvalue for the autocorrelation matrix (9.244) for $E_b/N_0 = 10$ dB and $\beta = 0.5$ is 40.14 (found with the aid of the MATLAB program `eig`) giving $0 < \mu < 0.05$; for $E_b/N_0 = 12$ dB the maximum eigenvalue is 63.04 giving $0 < \mu < 0.032$. A plot of the bit error probability versus E_b/N_0 for the three-tap equalizer with adaptive weights is compared with that for the unequalized case in Figure 9.33. Note that the gain provided by equalization over the unequalized case is over 2.5 dB in E_b/N_0 .

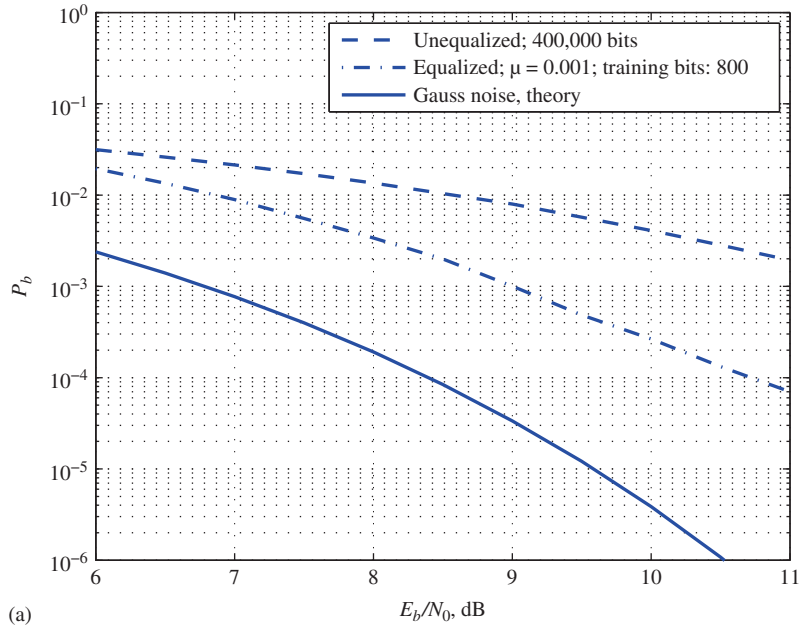


Figure 9.33

(a) Bit error probability plots for an adaptive MMSE equalizer. (b) Error and adaptation of weights.

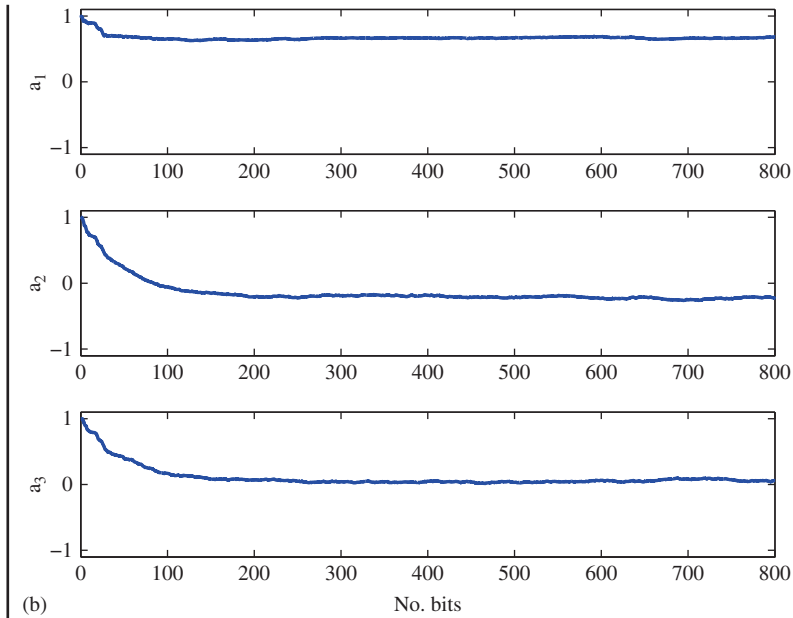


Figure 9.33
(Continued)

There are many more topics that could be covered on equalization, including decision feedback, maximum-likelihood sequence, and Kalman equalizers to name only a few.²⁶

Further Reading

A number of the books listed in Chapter 3 have chapters covering digital communications at roughly the same level as this chapter. For an authoritative reference on digital communications, see Proakis (2001).

Summary

1. Binary baseband data transmission in additive white Gaussian noise with equally likely signals having constant amplitudes of $\pm A$ and of duration T results in an average error probability of

$$P_E = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

where N_0 is the single-sided power spectral density of the noise. The hypothesized receiver is the integrate-and-

dump receiver, which turns out to be the optimum receiver in terms of minimizing the probability of error.

2. An important parameter in binary data transmission is $z = E_b/N_0$, the energy per bit divided by the noise power spectral density (single-sided). For binary baseband signaling, it can be expressed in the following equivalent forms:

$$z = \frac{E_b}{N_0} = \frac{A^2T}{N_0} = \frac{A^2}{N_0(1/T)} = \frac{A^2}{N_0B_p}$$

²⁶See Proakis, 2001, Chapter 11.

rest of the time they interfere constructively. The interference can be separated into intersymbol interference of the signaling pulses and cancellation due to the carriers of the direct and multipath components arriving out of phase.

10. Fading results from channel variations caused by propagation irregularities resulting in multiple transmission paths (termed multipath) through the communication medium. Fading results if the differential delays of the multipath components are short with respect to a symbol period but significantly long compared with the wavelength of the propagating signal. A commonly used model for a fading channel is one where the envelope of the received signal has a Rayleigh pdf. In this case, the power or symbol energy of the received signal can be modeled as having an exponential pdf, and the probability of error can be found by using the previously obtained error probability expressions for nonfading channels and averaging over the signal energy with respect to the assumed exponential pdf of the energy. Figure 9.30 compares the error probability for fading and nonfading cases for various modulation schemes. Fading results in severe degradation of the performance of a given modulation scheme. A way to combat fading is to use diversity.

11. Intersymbol interference results in a multipath channel having differential delays of the multipath components that are a significant fraction of a symbol period or even of the order of several symbol periods. Equalization can be used to remove a large part of the intersymbol interference introduced by multipath or channel filtering. Two techniques were briefly examined: zero-forcing and minimum-mean-squared error. Both can be realized by tapped delay-line filters. In the former technique, zero intersymbol interference is forced at sampling instants separated by multiples of a symbol period. If the tapped delay line is of length $(2N + 1)T$, where T is the symbol period, then N zeros can be forced on either side of the desired pulse. In a minimum mean-square-error equalizer, the tap weights are sought that give minimum mean-square error between the desired output from the equalizer and the actual output. The resulting weights for either case can be precalculated and preset, or adaptive circuitry can be implemented to automatically adjust the weights. The latter technique can make use of a training sequence periodically sent through the channel, or it can make use of the received data itself, assuming the error probability is fairly good, in order to carry out the minimizing adjustment.

Drill Problems

9.1 Given that $Q(\sqrt{2z}) = 10^{-6}$ for $z = 11.31$ find the following:

- (a) The signal amplitude (assume rectangular pulses) for antipodal baseband signaling that gives $P_E = 10^{-6}$ in a channel with $N_0 = 10^{-7}$ W/Hz and a data rate of 1 kbps.
- (b) Same question as in (a), but for a data rate of 10 kbps.
- (c) Same question as in (a), but for a data rate of 100 kbps.
- (d) The signal amplitude for antipodal baseband signaling that gives $P_E = 10^{-6}$, but for $N_0 = 10^{-5}$ W/Hz and a data rate of 1 kbps.
- (e) Same question as in (d), but for a data rate of 10 kbps.
- (f) Same question as in (d), but for a data rate of 100 kbps.

9.2 Taking the required channel bandwidth to be the first null of the rectangular pulse representing a 1 or 0 (plus for 1 and minus for 0), give bandwidths for the following data rates for a baseband transmission system using nonreturn-to-zero encoding:

- (a) 100 kbps;
- (b) 1 Mbps;
- (c) 1 Gbps;
- (d) 100 kbps but split phase encoded;
- (e) 100 kbps using nonreturn-to-zero encoding, but translated to a carrier frequency of 10 MHz.

9.3 Consider PSK with 10% of the transmitted signal power in a carrier component.

- (a) What is the value of m for the transmitted signal?
- (b) What is the change in phase in degrees each time the data switches?
- (c) If the total $E_b/N_0 = 10$ dB (including power dedicated to the carrier), what is P_E ?
- (d) If the total E_b/N_0 of 10 dB were available for data, what would the P_E be?

9.4

- (a) Rank the following binary modulation schemes in terms of E_b/N_0 required to give $P_E = 10^{-6}$ from best to worst (the lowest required E_b/N_0 is the best system): PSK; coherent FSK; DPSK; coherent ASK; and noncoherent FSK.
- (b) Do the same with respect to bandwidth efficiency.