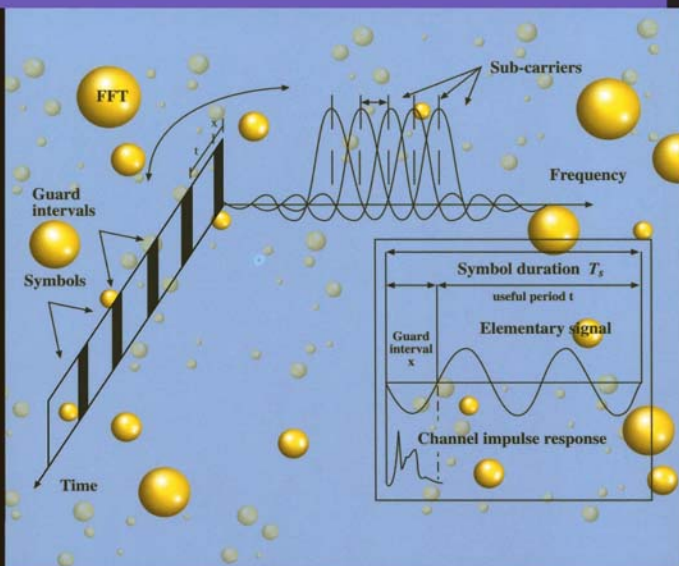


Multi-Carrier Digital Communications

Theory and Applications of OFDM

Second Edition



where M_j is the number of points in the QAM constellation, P_j is its power, N_j is the noise power in the that sub-channel after receiver processing, and K is the average number of its nearest neighbors, and is usually (but not necessarily) an integer.

Equation 3.15 is strictly applicable to square equiprobable constellations. However, very little error will result if we instead use the same analysis for more general equiprobable constellations. For example, a large QAM constellation with a circular boundary will have only 0.2 dB less average power than a square constellation with the same number of points and spacing between points. If further “shaping gain” is provided by using a multi-dimensional hyperspheric constellation boundary over several QAM sub-channels, then up to 1.53 dB average power reduction can be achieved. However, such shaping gain is rarely used in OFDM systems

$$p_{eb} = \frac{1}{\sum_j b_j} \sum_j b_j p_{ebj}. \quad (3.16)$$

For a complete OFDM system, the overall average bit error probability is

$$P_{ej} = 2 \frac{L-1}{L} Q \left(\frac{A}{\sqrt{N_j}} \right). \quad (3.17)$$

Clearly if the p_{ebj} are all equal, that will also be the overall error probability. Otherwise, the higher error probabilities will dominate performance. A sensible design goal is, therefore, to try to achieve the same error probabilities on all sub-channels if possible.

3.3 Bit Allocation

We now address the problem of optimizing the performance of an OFDM system over a stationary, non-flat, linear channel by choosing

the transmitted signal. We may either seek to maximize the overall bit rate with a required error probability, or we may minimize the error probability for a given bit rate. We will first deal with the former optimization, that is, the maximization of

$$R_b = \frac{1}{T} \sum_j b_j, \quad (3.18)$$

where b_j is the number of bits carried by the constellation of the j^{th} sub-carrier. We will deal primarily with the case of high signal-to-noise ratio (low error probability).

The total transmitted power is assumed to be constrained. We assume that the multi-carrier signal consists of a large number of non-interfering sub-channels, separated in frequency by $\delta = 1/T$. δf is assumed to be small enough so that the channel gain and the noise power spectral density are constant over each of the sub-channels. So, the channel gain and noise power on the j^{th} sub-channel can be given by H_j and N_j respectively. It is assumed that the noise is Gaussian. This assumption may seem questionable when the noise is primarily due to crosstalk, as on most cabled wire-pair media. However, the assumption becomes valid when there are many such interferers, and/or when they are passed through a narrow filter, as is the case in OFDM.

The variables in the optimization are the individually transmitted sub-channel powers P_j and the individual constellation sizes M_j . Fortunately, these optimizations are separable. We may first optimize the power distribution, under the constraint

$$\sum_j P_j = P. \quad (3.19)$$

It would appear that the constraint of an overall error probability is best met by requiring the same error probability for each of the sub-channels. This is indeed true at low error probabilities, which is the condition under study here, however it has been shown in general

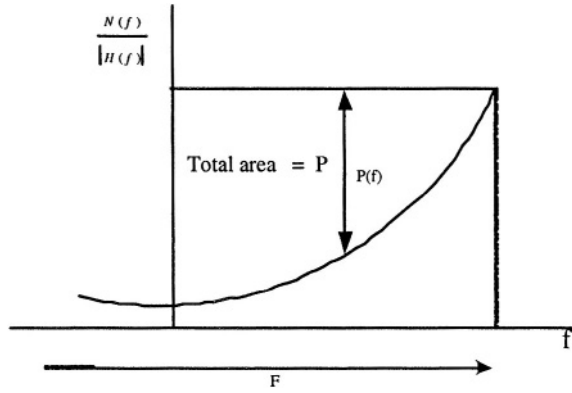


Figure 3.1: Optimum transmit power distribution by the water pouring theorem

and λ is the value for which

$$\int_F P(f) df = P. \quad (3.25)$$

This is illustrated in Figure 3.1. The curve of $N(f)/|H(f)|^2$ may be thought of as a bowl into which water whose area is equal to the allotted power is poured. The level of water is λ , and the distance between that level and the curve is the optimum transmit power spectrum. The range of F has been shown as contiguous for simplicity, but, in general, this need not be true.

It was shown in [21] that the optimum multi-carrier spectrum, under the assumptions given above, has a very similar form:

$$p_j = \begin{cases} \lambda - \frac{N_j \Gamma^2}{3P|H_j|^2}, & j \in F \\ 0, & j \in F' \end{cases} \quad (3.26)$$

where

$$F = \{j : \frac{N_j \Gamma^2}{3P|H_j|^2} \leq \lambda\}, \quad (3.27)$$

and λ is the value that causes

$$\sum_j P_j = P. \quad (3.28)$$

Note that over a region for which $N_j/|H_j|^2$ is small, P_j is approximately constant $\approx \lambda$. In a practical system with a low required error probability, we delete the region where $N_j/|H_j|^2$ is not low enough to support a sub-carrier with at least 4 constellation points at the required error probability. The result is an approximately flat spectrum over a region somewhat narrower than F . It was shown in [21] and verified in [22] that using a flat transmit spectrum results in very little degradation over the optimum when the required error probability is low.

A near-optimum design of a multi-carrier system therefore is to set all sub-carriers which cannot support a minimum size constellation to zero, and then, at least initially, to divide the allotted transmit power equally among the remaining sub-carriers. The next problem is to choose the constellation sizes of the non-zero sub-carriers.

Firstly, we will treat the constellation sizes as being continuous variables. Restricting M_j to an integer, which of course is essential, has almost no effect on the result. On the other hand, restricting $b_j = \log_2 M_j$ to an integer, which is convenient but not essential, does have a significant effect. Solving Equation 3.21,

$$M_j = 1 + \frac{3P_j|H_j|^2}{N_j\Gamma^2} \quad (3.29)$$

Rounding down to the next lower integer has little effect, and we have now found an approximation to the problem of maximizing the overall bit rate under an error probability constraint. The resultant bit rate is

$$R_b = \frac{1}{T} \sum_j \log_2 \left[1 + \frac{3P_j |H_j|^2}{N_j \Gamma^2} \right] \quad (3.30)$$

Since the sub-channels are closely spaced, the above may be closely approximated by the integral

$$R_b = \int_F \log_2 \left[1 + \frac{3P(f) |H_j|^2}{N(f) \Gamma^2} \right] df. \quad (3.31)$$

The above is the same as the well-known formula for the capacity of the channel, in which the signal-to-noise ratio,

$$\rho(f) = \frac{P(f) |H_j|^2}{N(f)} \quad (3.32)$$

has been decreased by a factor of $\frac{3}{\Gamma^2}$. This factor has been referred to as the “gap” [23]. In other words, the actual rate achievable, at a given error probability, on a given channel by an uncoded multi-carrier system is equal to the capacity of the same channel in which the signal-to-noise ratio has been reduced by the gap.

The above integral (Equation 3.31), except for a small change in the range of integration, is the same as the rate achievable over the same channel by a single carrier system using an ideal decision feedback equalizer (DFE). This leads to the very interesting result that at low error probability, the performance of an optimized multi-carrier system is approximately the same as that of a single carrier system with an ideal DFE [25]. This is particularly true when compared with an MMSE DFE as opposed to a zero-forcing DFE [26]. At higher error probabilities, the multi-carrier system has been shown to perform somewhat better [19].

We can rewrite Equation 3.30 as:

$$R_b = \frac{1}{T} \sum_j \log_2 \left[1 + \frac{3\rho_j}{\Gamma^2} \right] = \frac{1}{T} \log_2 \left\{ \prod_j \left[1 + \frac{3\rho_j}{\Gamma^2} \right] \right\}. \quad (3.33)$$

Now define an average signal-to noise ratio [23] such that

$$R_b = \frac{1}{T} \log_2 \left[1 + \frac{3\bar{\rho}}{\Gamma^2} \right], \quad (3.34)$$

Then, with n the number of sub-channels,

$$\bar{\rho} = \frac{\Gamma^2}{3} \left\{ \left(\prod_{j=1}^n \left[1 + \frac{3\rho_j}{\Gamma^2} \right] \right)^{\frac{1}{n}} - 1 \right\} \quad (3.35)$$

and at high signal-to-noise ratio,

$$\bar{\rho} = \left[\prod_{j=1}^n \rho_j \right]^{\frac{1}{n}}. \quad (3.36)$$

Thus, at high signal-to-noise ratio, when a multi-carrier system is fully optimized, the average signal-to-noise ratio is the geometric mean of the signal-to-noise ratios of the sub-channels. This is an important quantity, and serves as a good quality measure of the channel. The integral form of the above appears in analysis of single carrier systems with DFE.

So far we have not required that the number of bits carried by each sub-carrier be an integer. Such non-integral values can be achieved by treating more than one sub-carrier as a constellation in a higher-dimensional space, and assigning an integral number of bits to that higher-dimensional constellation. However, to avoid that complexity, it is usually required that b_j be an integer. Simply performing the above optimization and rounding each b_j down results in the average loss of $\frac{1}{2}$ bit per sub-carrier.

The above loss can be mostly recovered by small adjustments in the sub-carrier powers. Those sub-carriers that required a large round-down can have their powers increased slightly in order to achieve one more bit. Those with a smaller round-down can have their power reduced while keeping the same number of bits. During this process of

power re-allocation, it is important to ensure that the constraint of total power is kept. After the process is completed, the power deviations will vary over a 3 dB range.

A further small increase in bit rate can be achieved by accounting for the bit error probability of each sub-carrier rather than the symbol error probability, which is higher. This involves calculating the bit error probability for each sub-carrier after the system has been optimized for symbol error probability. Those sub-carriers which can meet the bit error probability target with an addition bit can then have their constellations increased accordingly.

3.4 Bit and Power Allocation Algorithms for Fixed Bit Rate

A more common design requirement is to minimize the bit error probability for a fixed bit rate, where that error probability must be below some target value. Again, we will assume a transmitter power constraint, and integral number of bits per sub-channel. Several algorithms have been developed to solve this design problem, with varying degrees of precision, complexity, and computation time [26]. One approach is similar to the previously described procedure for maximizing a variable bit rate. As before, we first determine Γ such that $Q(\Gamma) = p/4$, where p is the desired error probability, and is conservatively chosen as the maximum symbol error probability for each sub-carrier. We now examine if the un-quantized total bit rate

$$R_b = \frac{1}{T} \sum_j \log_2 \left[1 + \frac{3P_j |H_j|^2}{N_j \Gamma^2} \right] \quad (3.37)$$

is greater than the required rate. If not, the desired performance cannot be achieved. Otherwise, if $R_b - B \geq \frac{A}{T} J$, where B is the required bit rate, A is a positive integer, and J is the number of used sub-carriers,

then A bits are subtracted from each sub-carrier. This step adds 3 dB of noise margin.

After the above procedure, the number of bits on each sub-carrier is rounded down to the next lowest integer. The resultant bit rate R' is compared with B . If $R' > B$, then a set of sub-carriers with the smallest round-off are each reduced by an additional bit. If $R' < B$, then a set of sub-carriers with the largest round-off are each increased by one bit. Finally, the powers of the sub-carriers are adjusted as before to achieve the same error probability for each.