

GEL7114 Digital communications

LTIS Linear Time-Invariant Systems

Module 1

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Topics covered

Energy signals and power signals	Sec. 1.2	pp. 14-16
Spectral density and self-correlation function	Sec. 1.3 & 1.4	pp. 16-20
Random processes	Sec. 1.5	pp. 20-30
Noise (AWGN)	Sec. 1.5.5	pp. 30-33
Ideal linear systems and filters	Sec. 1.6	pp. 33-44
Bandwidth	Sec. 1.7	pp. 45-50

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Linearity

➤ Defining a linear system

- A linear combination of inputs generates the same linear combination of outputs

$$\begin{aligned} x_1(t) &\Rightarrow y_1(t) & ax_1(t) + bx_2(t) &\Rightarrow ay_1(t) + by_2(t) \\ x_2(t) &\Rightarrow y_2(t) \end{aligned}$$

➤ Example - ideal amplifier

$$y(t) = Rx(t)$$

➤ Counterexample

$$y(t) = x^2(t)$$

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Time invariance

- An input lag causes the same lag for the output

$$x_1(t) \Rightarrow y_1(t) \quad x_1(t-\tau) \Rightarrow y_1(t-\tau)$$

➤ Example

$$y(t) = Rx(t)$$

➤ Counterexample

$$y(t) = x(t)\sin t$$

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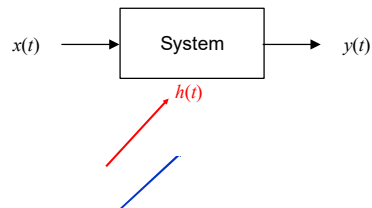
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LTIS (Filtering)

- Linearity and time invariance

➤



- Impulse response
- Frequency response

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Input/output relationship

- Filter: frequency response

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

- Impulse response

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega) e^{j\omega t} d\omega$$

- Input/output relationship *in time domain*

$$y(t) = \{h * x\}(t)$$

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Convolution

- Convolution of $f(t)$ and $g(t)$

$$\{f * g\}(t) = \int_{-\infty}^{+\infty} f(u)g(t-u)du$$

- Identity element for convolution $\delta(t)$

$$\begin{aligned}\{f * g\}(t) &= \int_{-\infty}^{+\infty} f(u)g(t-u)du \\ &= \int_{-\infty}^{+\infty} \delta(u)g(t-u)du = g(t) \int_{-\infty}^{+\infty} \delta(u)du = g(t)\end{aligned}$$

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Impulse response

- Delta function - identity element for convolution
- Origin of the expression "impulse response"
 $y(t) = \{h * \delta\}(t) = h(t)$
- output when entry is a Dirac pulse (delta function)



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Parseval's theorem

$f(t)$ a function of $L^2(\mathbb{R})$ *energy function*

total energy $\rightarrow E = \int_{-\infty}^{+\infty} |f(t)|^2 dt$

Energy Spectrum

$$E(\omega) = \frac{1}{2\pi} |F(\omega)|^2$$

Energy in a frequency band

$$E(\omega_1 \leq \omega \leq \omega_2) = \int_{\omega_1}^{\omega_2} E(\omega) d\omega = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

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Spectral density at the output of a filter

- LTIS with frequency response $H(f)$
- Spectral density at the input $G_X(f)$
- Spectral density at the output

$$G_Y(f) = G_X(f) |H(f)|^2$$

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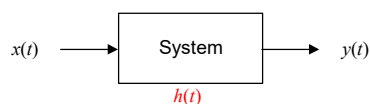
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Response to a cosine

- Input signal $x(t) = \cos(2\pi f_0 t)$



$$y(t) = |H(f_0)| \cos(2\pi f_0 t + \text{Arg}(H(f_0)))$$

- Output is also a cosine

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LTIS – bandwidth conservation

- Filter $Y(f) = H(f)X(f)$
 ☐ no new frequencies can be produced
- Band-limited input $\Rightarrow$
 Band-limited output

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Ideal filtering

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Causality

➤ Causal system

- At a specific time, the output is only a function of the input at that time or before (but never after)

➤ Example

$$y(t) = Rx(t) \quad y(t) = x(t) \sin t \quad y(t_0) = \frac{1}{C} \int_{-\infty}^{t_0} x(z) dz$$

➤ Counterexample

$$y(t_0) = \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} x(z) dz$$

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Ideal transmission line

➤ Undistorted

$$y(t) = Kx(t - t_0)$$

➤ Frequency response

$$H(f) = Ke^{-j2\pi ft_0}$$

$$|H(\omega)| = K \quad \text{Arg } H(f) = 2\pi ft_0 = f / f_0$$

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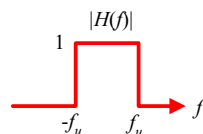
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Ideal low pass filter

$$H_{LP}(f) = \text{Rect}\left(\frac{f}{2f_u}\right)$$



- low-frequency components pass without change
- high-frequency components are zeroed out

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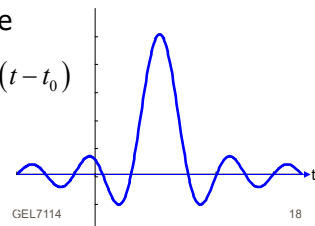
Ideal low pass filter

➤ Frequency response

$$H(f) = \text{Rect}\left(\frac{f}{2f_u}\right)$$

➤ Impulse response

$$h(t) = 2f_u \text{sinc } 2f_u(t - t_0)$$



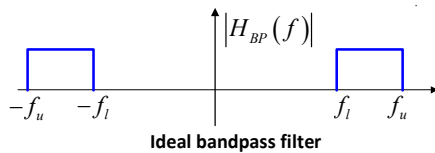
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Ideal bandpass filter



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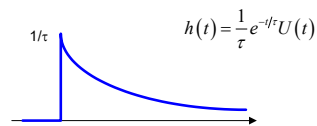
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RC low pass filter

- Frequency response

$$H(f) = \frac{1}{1 + j2\pi f\tau}$$

- Impulse response



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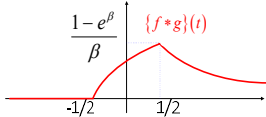
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Output for Rect input

- Output is convolution of input with impulse response

$$y(t) = \{h * x\}(t) = \begin{cases} 0 & \text{pour } t \leq -1/2 \\ \tau(1 - e^{-(t+1/2)/\tau}) & \text{pour } -1/2 \leq t \leq 1/2 \\ 2\tau sh(1/2\tau)e^{-t/\tau} & \text{pour } t \geq 1/2 \end{cases}$$



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Causality and filtering

- Definition

□ Output at some point is only a input function at this time or before (not after)

- Impulse response

$$y(t) = \{h * x\}(t) = \int_{-\infty}^{\infty} x(u)h(t-u)du = \int_{-\infty}^{t'} x(u)h(t-u)du$$

Causality

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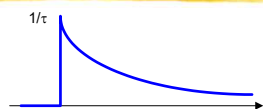
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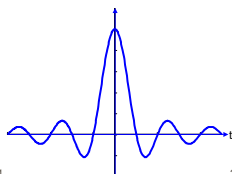
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Example: low pass filters

- Causal
 - RC low pass filter



- Non-causal
 - Ideal low pass filter



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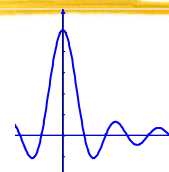
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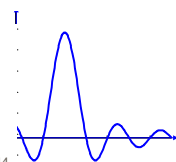
Make a filter causal

- Truncate



- Introduce a delay

$$h_{causal}(t) = h(t-\tau)U(t)$$



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Bandwidth definitions

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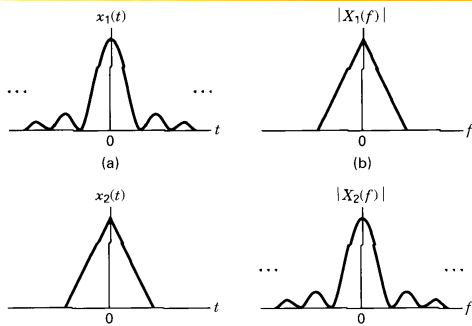
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Band-limited signal



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Band-limited signal

- A time-limited signal cannot be band-limited
⇒ there are no band-limited signals
- A signal with a spectrum with a steep roll-off is "less wide" than a signal with a spectrum that drops less quickly in frequency...
- How to quantify bandwidth ?

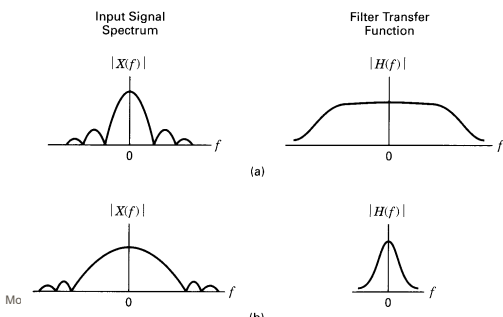
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Filtering and bandwidth



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Bandwidth

- Absolute bandwidth
- 3 dB bandwidth (Half power)
- Noise equivalent
- Null-to-Null
- Fractional power
- PSD bound

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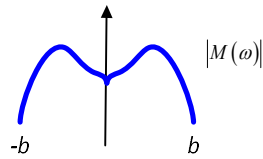
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Absolute bandwidth

- $|\omega| \geq b \Rightarrow M(\omega) = 0$



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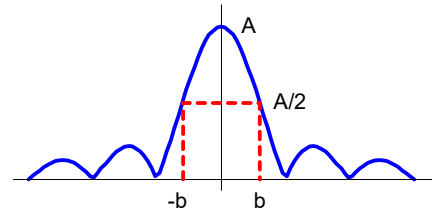
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3 dB bandwidth

- The bandwidth when the spectrum reaches half its maximum value



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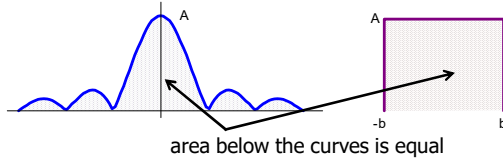
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Noise equivalent

- The bandwidth of a rectangular spectrum that has the same maximum value and the same total energy



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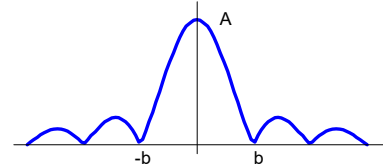
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Null-to-Null

- The bandwidth between the first zeros around the maximum value



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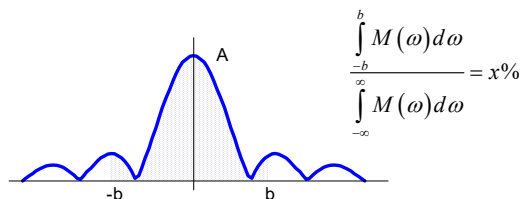
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Fractional power

- The bandwidth where x% of the total energy is found



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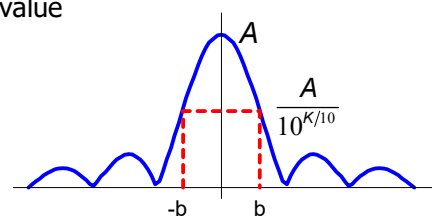
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PSD bound

- The bandwidth where the spectrum decreased by K db from the maximum value



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Graph

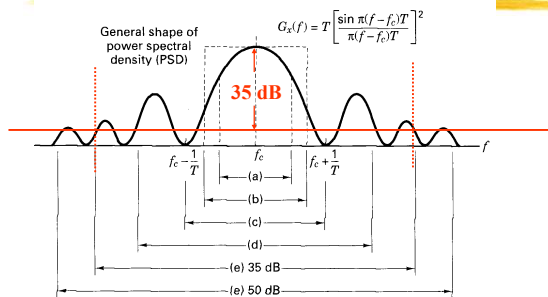


Figure 1.20 Bandwidth of digital data. (a) Half-power, (b) Noise equivalent, (c) Null to null, (d) 99% of power, (e) Bounded PSD (defines attenuation outside bandwidth) at 35 and 50 dB.

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ESD, PSD, autocorrelation function

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Energy function

$f(t)$ a function in $L^2(\mathcal{R})$ energy function

$$\text{Total energy} \rightarrow E = \int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{+\infty} E(\omega) d\omega$$

$$\text{energy per bit} \rightarrow E_b = \int_{-T_b/2}^{+T_b/2} |f(t)|^2 dt$$

$$\text{autocorrelation} \rightarrow R(\tau) = \int_{-\infty}^{+\infty} f(t)f(t-\tau) dt$$

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Autocorrelation function energy signals

Definition

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau) dt \quad \text{for } -\infty < \tau < \infty$$

Properties

- $R_x(\tau) = R_x(-\tau)$
- $|R_x(\tau)| \leq R_x(0)$ for all τ
- $R_x(\tau) \leftrightarrow \psi_x(f)$
- $R_x(0) = \int_{-\infty}^{\infty} x^2(t) dt$

symmetrical in τ about zero
maximum value occurs at the origin
autocorrelation and ESD form a Fourier transform pair, as designated by the double-headed arrows
value at the origin is equal to the energy of the signal

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Autocorrelation function energy signals

Definition

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau) dt \quad \text{for } -\infty < \tau < \infty$$

Properties

- $R_x(\tau) = R_x(-\tau)$
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$$TF\{R(c)\} = TF\left\{\int_{-\infty}^{\infty} f(t)f(t+c)dt\right\}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)f(t+c)dt e^{-j\omega c} dc \\ &= \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} f(t+c) e^{-j\omega c} dc dt \\ &= \int_{-\infty}^{\infty} f(t) e^{j\omega t} \int_{-\infty}^{\infty} f(t+c) e^{-j\omega c} e^{-j\omega t} dt dc \\ &= \int_{-\infty}^{\infty} f(t) e^{j\omega t} \int_{-\infty}^{\infty} f(z) e^{-j\omega z} dz dt \\ &= \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt \cdot \int_{-\infty}^{\infty} f(z) e^{-j\omega z} dz \\ &= F^*(\omega) \cdot F(\omega) = |F(\omega)|^2 = E(\omega) \end{aligned}$$

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Energy vs. Power

$$E = \lim_{T \rightarrow \infty} \int_{-T/2}^{+T/2} |f(t)|^2 dt$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2 dt$$

$$\text{> } E < \infty \Rightarrow P$$

$$\text{> } P < \infty \Rightarrow E = \infty \text{ power signal}$$

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Energy vs. Power

$$E = \lim_{T \rightarrow \infty} \int_{-T/2}^{+T/2} |f(t)|^2 dt$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2 dt$$

$$\text{> } E < \infty \Rightarrow P = 0 \text{ energy signal}$$

$$\text{> } P < \infty \Rightarrow E = \infty \text{ power signal}$$

- ☐ All realistic signals are energy signals
- ☐ All periodic signals are power signals
- ☐ All random signals are power signals

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Periodic signals

$$\text{> Fourier series representation}$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

$$\text{> PSD}$$

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - n f_0)$$

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Autocorrelation function periodic signals

$$\text{> Definition}$$

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x(t+\tau) dt$$

$$\text{> Properties}$$

1. $R_x(\tau) = R_x(-\tau)$ symmetrical in τ about zero
2. $|R_x(\tau)| \leq R_x(0)$ for all τ maximum value occurs at the origin
3. $R_x(\tau) \leftrightarrow G_x(f)$ autocorrelation and PSD form a Fourier transform pair
4. $R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t) dt$ value at the origin is equal to the average power of the signal

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Energy vs. Power

$$E = \lim_{T \rightarrow \infty} \int_{-T/2}^{+T/2} |f(t)|^2 dt$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2 dt$$

$$\text{> } E < \infty \Rightarrow P = 0 \text{ energy signal}$$

$$\text{> } P < \infty \Rightarrow E = \infty \text{ power signal}$$

- ☐ All realistic signals are energy signals
- ☐ All periodic signals are power signals
- ☐ All random signals are power signals

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Random processes

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PSD, Random processes, AWGN

- Random signals
 - Random variable vs. random process
 - Stationarity - ergodicity
 - Examples
 - PSD
- Additive White Gaussian Noise (AWGN)

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Random variables

- Distribution function $F_X(x)$
- Probability density $f_X(x)$
- Mathematical expectation of a random variable

$$EX = \int_{-\infty}^{\infty} xf_X(x)dx$$

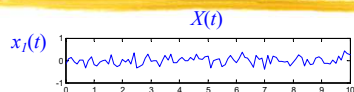
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Random (stochastic) processes



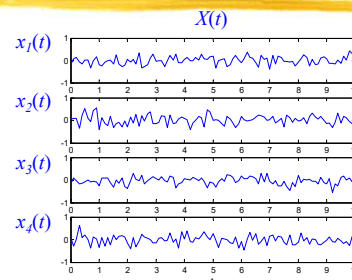
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Random (stochastic) processes



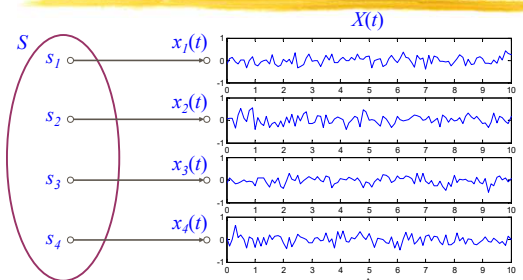
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Random processes (stochastic)



$$x_i(t) = X(t, s_i), \text{ with } s_i \in S$$

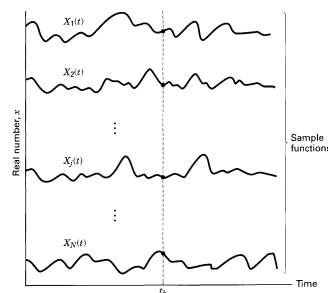
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Random variable vs. random process

process $X(t)$

variable $X(t_k)$

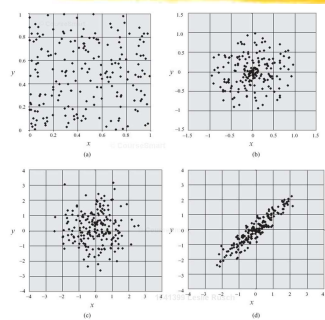


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Figure 1.5 Random noise process.

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Two random variables



Scatterplots
visualization of
relative frequency

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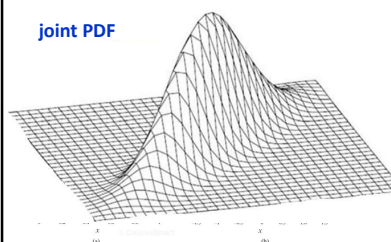
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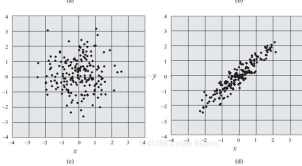
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joint PDF

es



Scatterplots
visualization of
relative frequency



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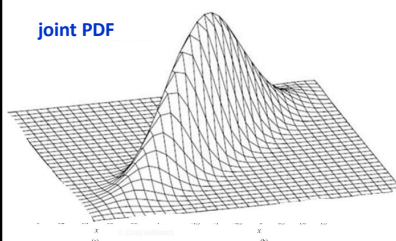
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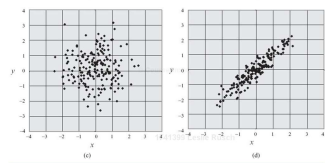
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joint PDF

es



Scatterplots
visualization of

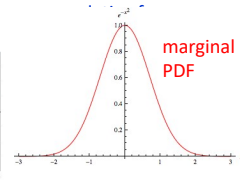


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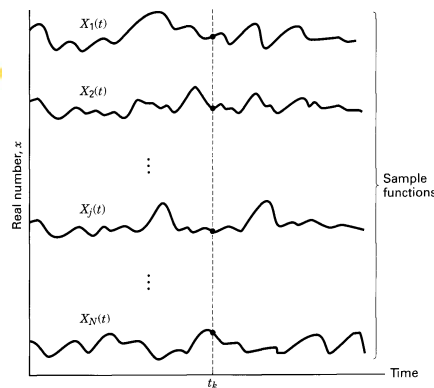
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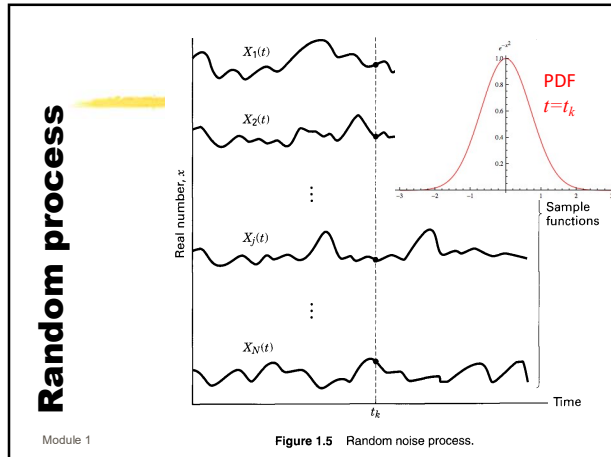
Random process



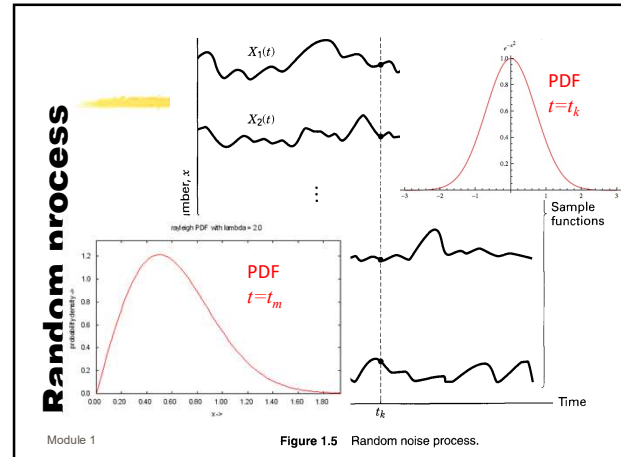
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Figure 1.5 Random noise process.

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Random processes

- Random, so a PDF; a function of time, possible correlations
 - Joint PDFs ...
 - Perhaps extremely complex
- Simplify by considering only 2 moments
- Expectation (mean)
 - autocorrelation
- Simplify again
 - only processes that are wide sense stationary (WSS) ...

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Expectation of a random process

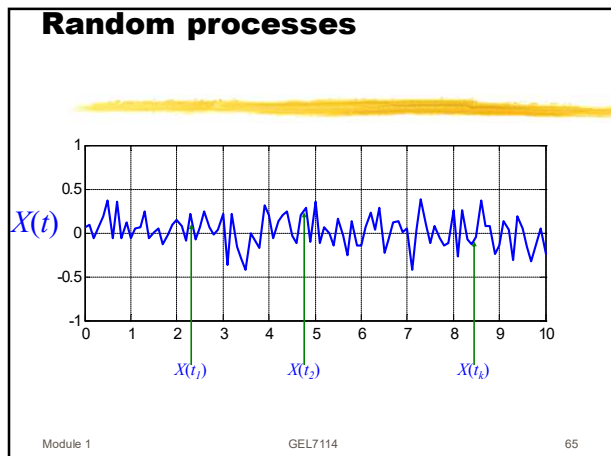
$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx$$

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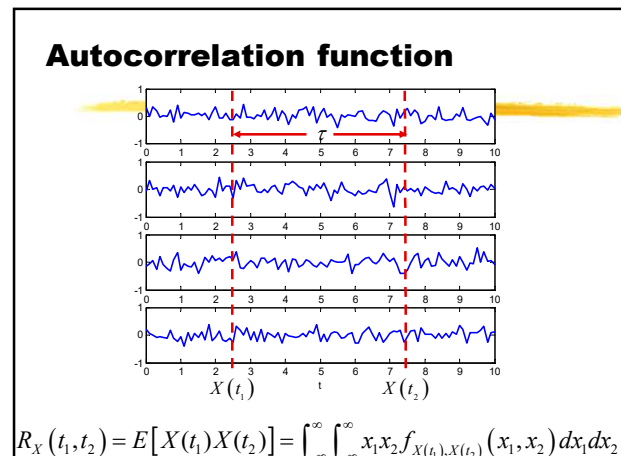
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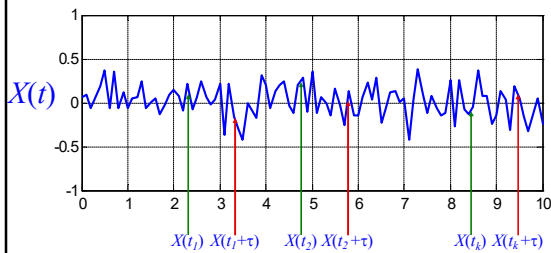


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Random processes



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Digital communications

Wide Sense Stationary (WSS) Random processes

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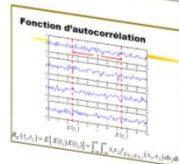
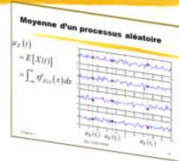
Wide sense stationary (WSS)

- The average is constant in time

$$\mu_X(t) = m_X$$

- Autocorrelation function is only a function of the time lag

$$R_X(t_1 - t_2) = R_X(\tau)$$



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Autocorrelation function

WSS stochastic signals

wide sense stationary

- Definition

$$R_X(\tau) = \mathbb{E}\{X(t)X(t + \tau)\} \quad \text{for } -\infty < \tau < \infty$$

- Properties

- $R_X(\tau) = R_X(-\tau)$ symmetrical in τ about zero
- $|R_X(\tau)| \leq R_X(0)$ for all τ maximum value occurs at the origin
- $R_X(\tau) \leftrightarrow G_X(f)$ autocorrelation and power spectral density form a Fourier transform pair
- $R_X(0) = \mathbb{E}\{X^2(t)\}$ value at the origin is equal to the average power of the signal

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PSD of a WSS process

- $G_X(f) \geq 0$ and is always real valued
- $G_X(f) = G_X(-f)$ for $X(t)$ real-valued
- $G_X(f) \leftrightarrow R_X(\tau)$ PSD and autocorrelation form a Fourier transform pair
- $P_X = \int_{-\infty}^{\infty} G_X(f) df$ relationship between average normalized power and PSD

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Ergodicity

- Time average

$$\mu_X^t(T) = \frac{1}{2T} \int_{-T}^T X(t, s_i) dt = \frac{1}{2T} \int_{-T}^T x_i dt$$

- Stochastic average, WSS process

$$\mathbb{E}[X(t)] = \mu_X$$

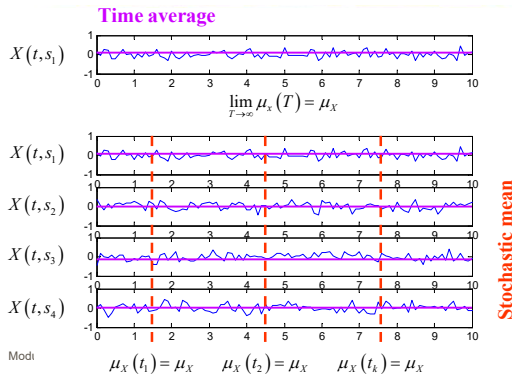
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Average ergodic random process



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Ergodicity

- a random process is **ergodic** if the time averages give the same results as the distribution averages
 - Example: average time = ensemble average
- Stationary $\Rightarrow$ ergodic
- Ergodic $\nRightarrow$ Stationary

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Equivalences

1. The quantity $m_X = E\{X(t)\}$ is equal to the dc level of the signal.
2. The quantity m_X^2 is equal to the normalized power in the dc component.
3. The second moment of $X(t)$, $E\{X^2(t)\}$, is equal to the total average normalized power.
4. The quantity $\sqrt{E\{X^2(t)\}}$ is equal to the root-mean-square (rms) value of the voltage or current signal.
5. The variance σ_X^2 is equal to the average normalized power in the time-varying or ac component of the signal.
6. If the process has zero mean (i.e., $m_X = m_X^2 = 0$), then $\sigma_X^2 = E\{X^2\}$ and the variance is the same as the mean-square value, or the variance represents the total power in the normalized load.
7. The standard deviation σ_X is the rms value of the ac component of the signal.
8. If $m_X = 0$, then σ_X is the rms value of the signal.

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WSS Example & counterexample

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example: WSS process

$$A \cos(2\pi f_c t + \Theta)$$

Sinusoid with random phase

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx$$

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phase Θ random, amplitude A constant

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example: WSS process

$$A \cos(2\pi f_c t + \Theta)$$

Sinusoid with random phase

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx$$

$$\mu_X(t) = E[A \cos(2\pi f_c t + \Theta)]$$

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example: WSS process

$$A \cos(2\pi f_c t + \Theta)$$

Sinusoid with random phase

$$\mu_X(t) = E[A \cos(2\pi f_c t + \Theta)]$$

$$\mu_X(t) = \int_{-\infty}^{\infty} A \cos(2\pi f_c t + \theta) f_{\Theta}(\theta) d\theta$$

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example: WSS process

$$A \cos(2\pi f_c t + \Theta)$$

Sinusoid with random phase

$$\mu_X(t) = E[A \cos(2\pi f_c t + \Theta)]$$

$$\mu_X(t) = \int_{-\infty}^{\infty} A \cos(2\pi f_c t + \theta) f_{\Theta}(\theta) d\theta$$

$$\mu_X(t) = \frac{A}{2\pi} \int_0^{2\pi} \cos(2\pi f_c t + \theta) d\theta$$

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example: processus WSS

$$A \cos(2\pi f_c t + \Theta)$$

Sinusoid with random phase

$$\mu_X(t) = E[A \cos(2\pi f_c t + \Theta)]$$

$$\mu_X(t) = \int_{-\infty}^{\infty} A \cos(2\pi f_c t + \theta) f_{\Theta}(\theta) d\theta$$

$$\mu_X(t) = \frac{A}{2\pi} \int_0^{2\pi} \cos(2\pi f_c t + \theta) d\theta$$

$$\mu_X(t) = 0$$

The average of the random process $X(t)$ is therefore constant, i.e., invariant over time.

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example: processus WSS

Suite ... Autocorrelation function

$$R_X(t_1, t_2) = E[A \cos(2\pi f_c t_1 + \Theta) A \cos(2\pi f_c t_2 + \Theta)]$$

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Wide sense stationary (WSS)

phase θ random, amplitude A constant

$$X(t) = A \cos(2\pi f_c t + \theta)$$

- The average is constant in time

$$\mu_X(t) = 0$$

$$\mu_X(t) = m_X$$

- Autocorrelation function is only a function of the time lag

$$R_X(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau)$$

$$R_X(t_1 - t_2) = R_X(\tau)$$

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counter-example: non-WSS process

Sinusoid with random amplitude

phase θ random, amplitude A constant

$$X(t) = A \cos(2\pi f_c t + \theta)$$

phase θ constant, amplitude A random

$$\mu_X(t) = E[X(t)] = E[A \cos(2\pi f_c t + \theta)]$$

$$\mu_X(t) = \int_{-\infty}^{\infty} a \cos(2\pi f_c t + \theta) f_A(a) da$$

$$\mu_X(t) = \cos(2\pi f_c t + \theta) \int_{-\infty}^{\infty} a f_A(a) da$$

$$\mu_X(t) = E[A] \cos(2\pi f_c t + \theta) = \mu_A \cos(2\pi f_c t + \theta)$$

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Not constant in time !!

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counter-example: non-WSS process
Sinusoid with random amplitude

phase θ random, amplitude A constant $X(t) = A \cos(2\pi f_c t + \theta)$
 phase θ constant, amplitude A random

$$\mu_X(t) = E[X(t)] = E[A \cos(2\pi f_c t + \theta)]$$

$$\mu_X(t) = E[A] \cos(2\pi f_c t + \theta) = \mu_A \cos(2\pi f_c t + \theta)$$

Not constant in time!!

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counter-example: non-WSS process
... Autocorrelation function

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_X(t_1, t_2) = E\{[A \cos(2\pi f_c t_1 + \theta)][A \cos(2\pi f_c t_2 + \theta)]\}$$

$$R_X(t_1, t_2) = E[A^2] \cos(2\pi f_c t_1 + \theta) \cos(2\pi f_c t_2 + \theta)$$

Not a function of $t_1 - t_2$!!

The random process $X(t)$ is not *wide sense stationary*.

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Power Spectral Density
Example

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Power Spectral Density

- PSD is the Fourier Transform of the autocorrelation function
- WSS random processes have an autocorrelation function of a single variable
- Quick example
 - ☐ random binary data
 - ☐ rectangular non-return to zero (NRZ)

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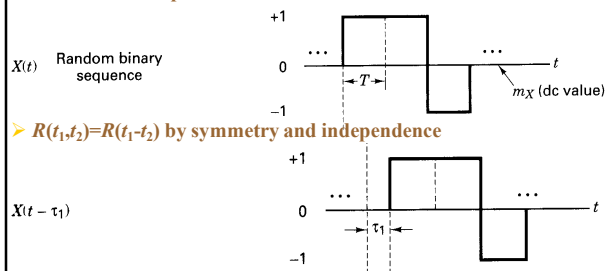
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Autocorrelation function
calculation

- The data is independent



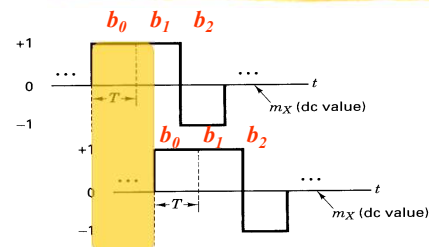
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Case 1 : $\tau > T$



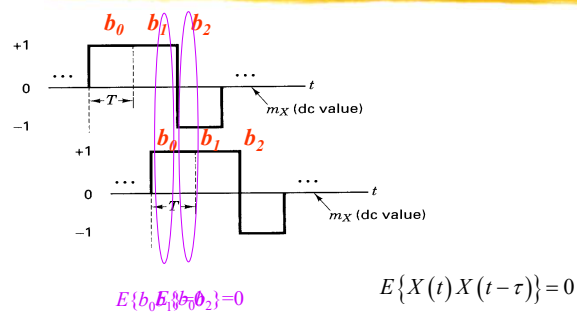
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Case 1 : $\tau > T$



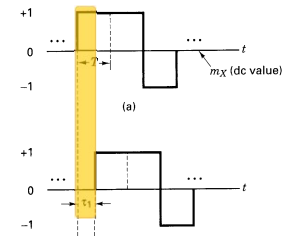
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Case 2 : $\tau < T$



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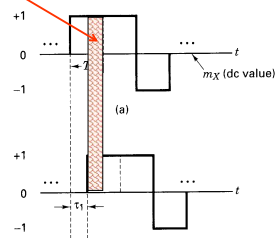
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Case 2 : $\tau < T$

- The data are the same τ to T



- The greater τ , the smaller the autocorrelation ...

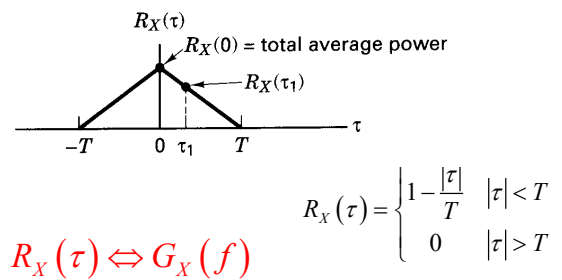
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Integration result



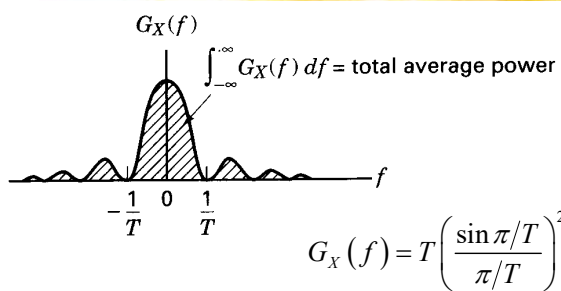
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PSD



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Additive White Gaussian Noise (AWGN)

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Topics covered

Energy signals and power signals	Sec. 1.2	pp. 14-16
Spectral density and self-correlation function	Sec. 1.3 & 1.4	pp. 16-20
Random processes	Sec. 1.5	pp. 20-30
Noise (AWGN)	Sec. 1.5.5	pp. 30-33
Ideal linear systems and filters	Sec. 1.6	pp. 33-44
Bandwidth	Sec. 1.7	pp. 45-50

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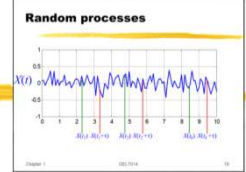


AWGN

- n is a sample of $n(t)$

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-n^2/2\sigma^2}$$

- zero mean, variance σ^2



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AWGN independent samples

- Samples are uncorrelated

$$E[n(t)n(t-\tau)] = \delta(\tau)$$

- For Gaussian variables, uncorrelated implies independent

$$f_{N_1 N_2}(n_1, n_2) = f_{N_1}(n_1) f_{N_2}(n_2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-n_1^2/2\sigma^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-n_2^2/2\sigma^2}$$

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PSD for AWGN

$$G_{noise}(f) \Leftrightarrow R_n(\tau) = \frac{N_0}{2} \delta(\tau)$$

$$G_{noise}(f) = \frac{N_0}{2} \text{ Watts/Hz}$$

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PSD for AWGN

$$G_{noise}(f) \Leftrightarrow R_n(\tau) = \frac{N_0}{2} \delta(\tau)$$

$$G_{noise}(f) = \frac{N_0}{2} \text{ Watts/Hz}$$

like white light...all frequencies present

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Filtered noise

- PSD at the output of the LTIS

$$G_Y(f) = G_X(f) |H_{LP}(f)|^2$$

- noise PSD at the input (AWGN) $G_X(f) = \frac{N_0}{2}$

- output PSD *colored noise* $G_Y(f) = \begin{cases} \frac{N_0}{2} & |f| < f_u \\ 0 & \text{elsewhere} \end{cases}$

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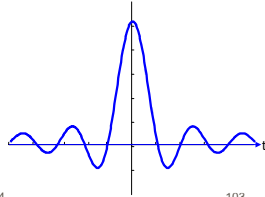
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Colored noise

- Noise samples are correlated
- Finite power
- Autocorrelation function

$$R_Y(\tau) = N_0 f_u \text{sinc}(2f_u \tau)$$

$$= N_0 f_u \frac{\sin 2\pi f_u \tau}{2\pi f_u \tau}$$



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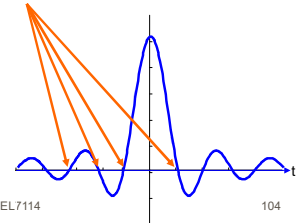
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Uncorrelated samples

- $R_Y(\tau)=0 \Rightarrow$ uncorrelated $\Rightarrow$ independent
- Samples at the zeroes of the sinc are not correlated
- Zeros:

$$\tau = \frac{n}{2f_u}$$



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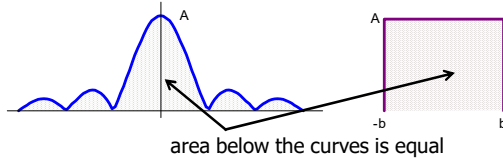
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Noise equivalent BW

- The bandwidth for a rectangular spectrum with the same maximum value and the same total energy



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Pulse Coded Modulation (PCM) versus Pulse Amplitude Modulation (PAM)

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Topics covered



Formatting analog information	Sec. 2.4	pp. 62-75
Baseband modulation waveforms	Sec. 2.8	pp. 85-94
Sources of corruption	Sec. 2.5	pp. 76-79

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Vocabulary

- bit
- symbol ← waveform
- alphabet or constellation

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Analog signals

- Sampling theorem
 - ❑ sample fast enough
 - ❑ perfect reconstruction is possible
 - ❑ assumes infinite resolution of sample
- Analog-to-Digital converters
 - ❑ finite resolution
 - ❑ finite bandwidth

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Digital data

- transmission media is analog
 - ❑ light on fiber
 - ❑ voltage on wire/cable
 - ❑ electric field on wireless
- transmitter
 - ❑ digital-to-analog conversion
 - ❑ achieve a symbol waveform that is "interesting"
 - ❑ finite resolution & bandwidth issues

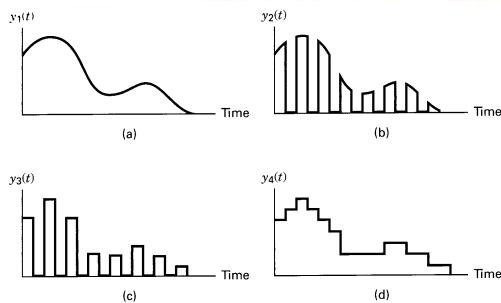
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Sampling analog signals



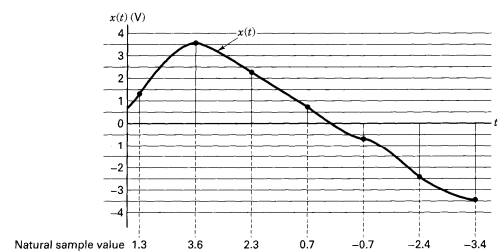
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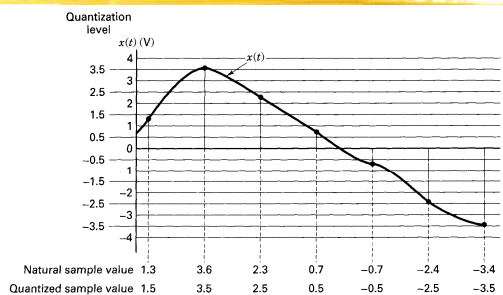
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Quantification



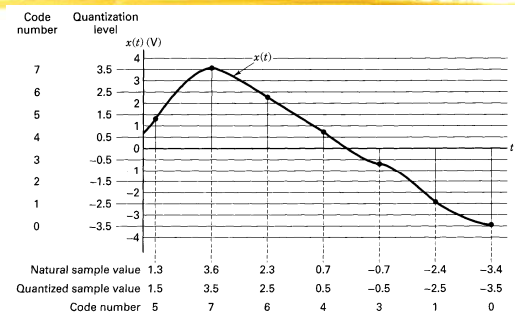
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Quantification

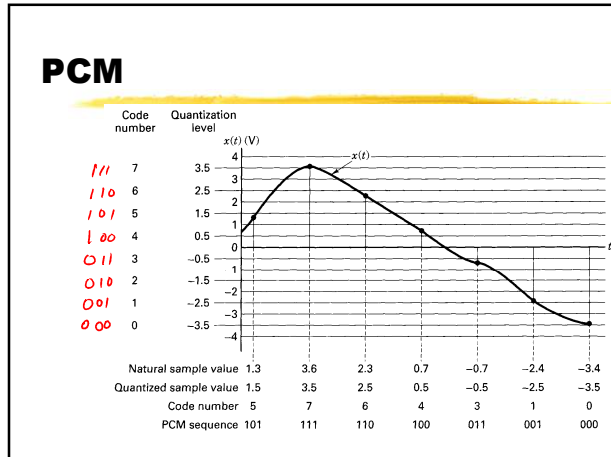


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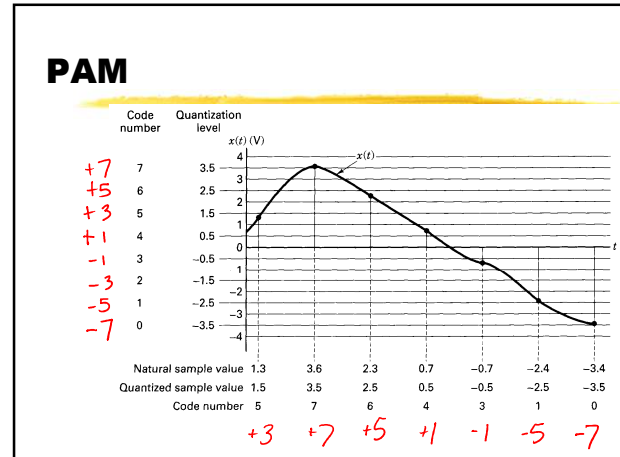
Coding



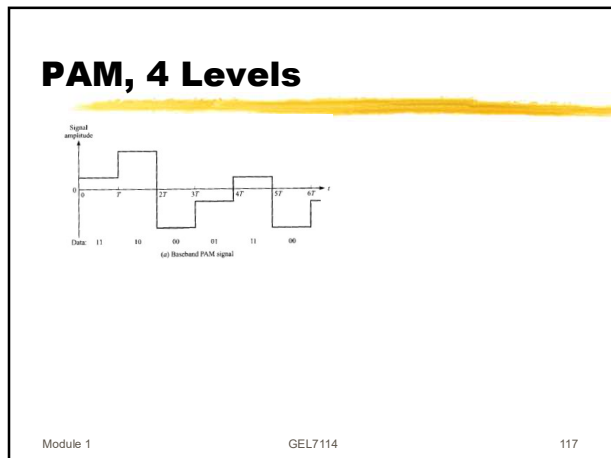
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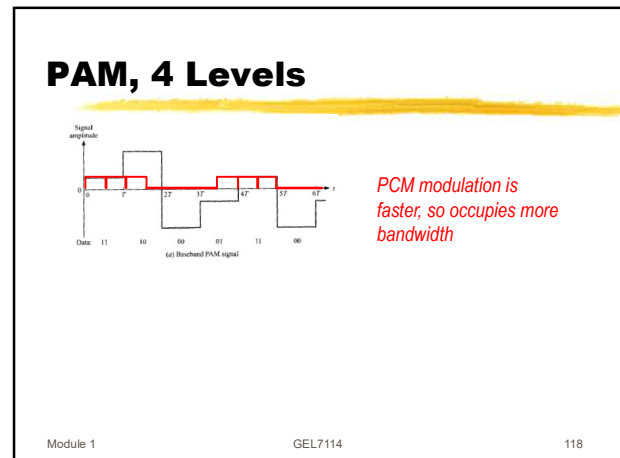
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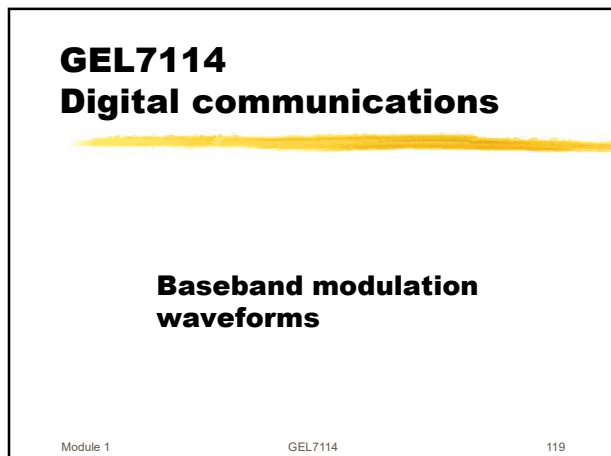
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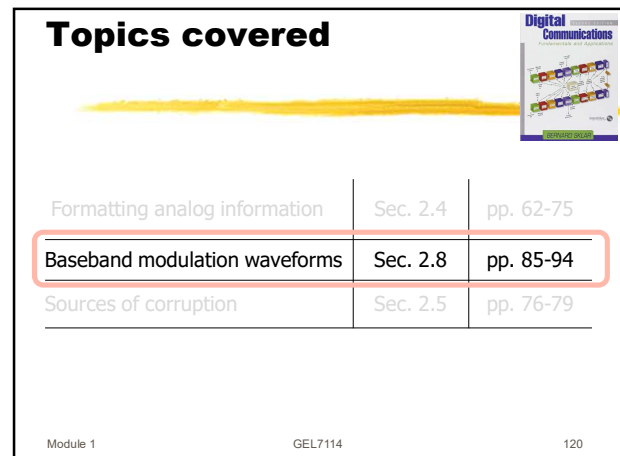
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PCM vs. PAM

- Pulse coded modulation
 - ☐ binary transmission
 - ☐ faster/wider bandwidth solution
- Pulse amplitude modulation
 - ☐ multilevel transmission
 - ☐ slower/narrower bandwidth solution

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RZ vs. NRZ

- Line codes
 - ☐ many varieties
 - ☐ two fundamental examples are RZ & NRZ
- Return-to-zero
 - ☐ "quiet" time between subsequent bits/symbols

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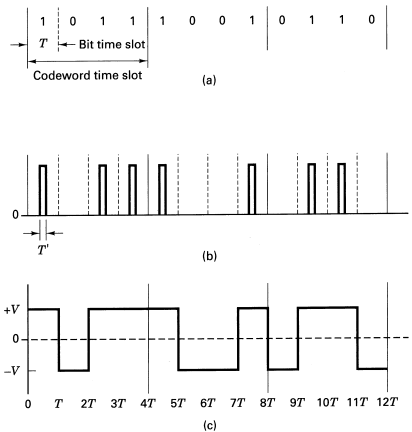
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Bit duration

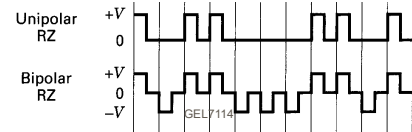
baseband transmission of PCM waveforms



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Line codes



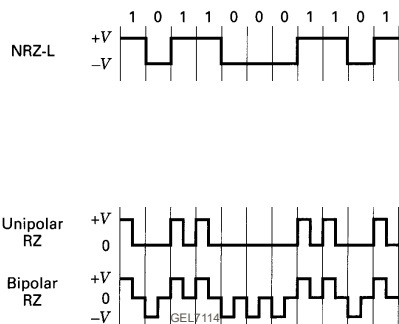
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Line codes



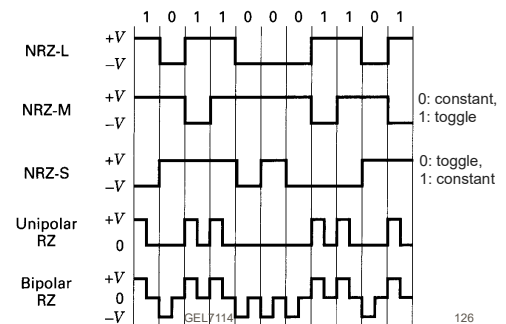
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Line codes



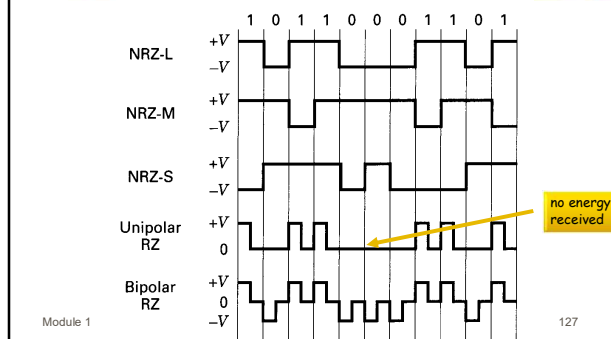
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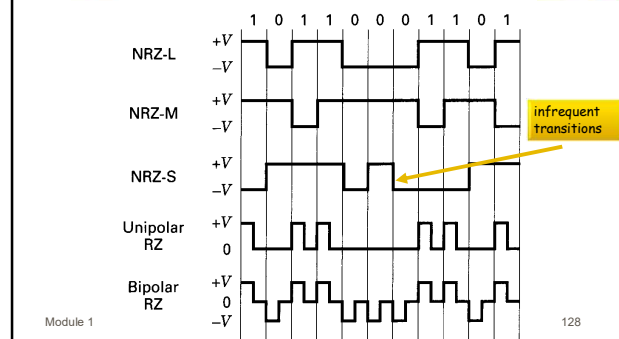
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Line codes



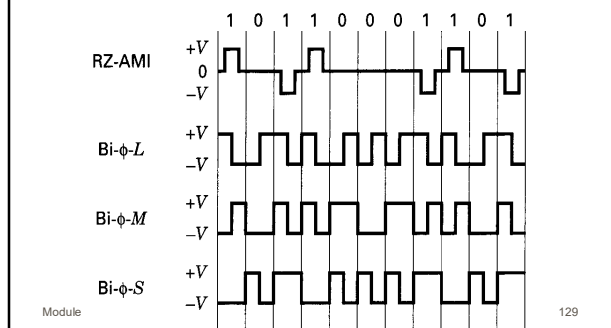
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Line codes



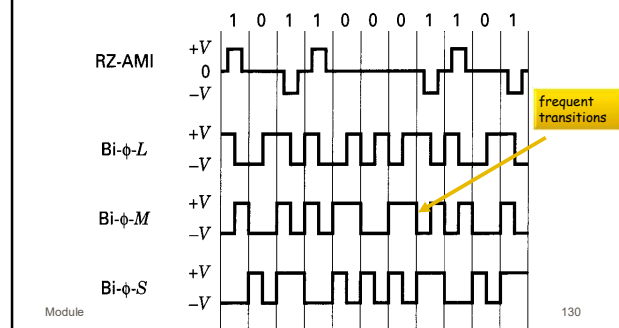
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Line codes



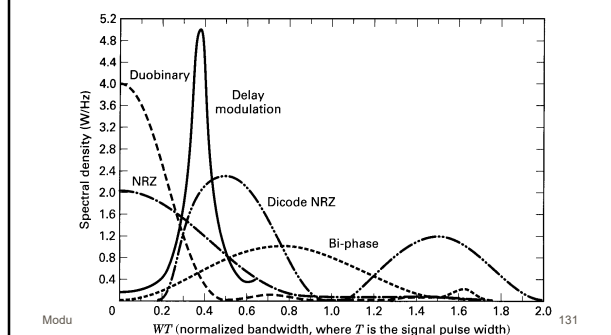
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Line codes



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Line code PSD



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Why this variety?

- DC Component
 - ☐ AC coupling supported, need for DC=0 dicode NRZ
- Clock recovery
 - ☐ frequent transitions RZ
 - ☐ guaranteed transitions Manchester, Bi-φ-M, etc.
 - ☐ error detection
- Bandwidth efficiency RZ vs. NRZ
- Easier detection
 - ☐ differential coding NRZ-M
- Noise rejection RZ vs. NRZ Unipolar vs. Bipolar

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Three criteria

- Is the communication link good?
 - ❑ Power efficiency
 - ❑ Spectral efficiency
 - ❑ Cost/complexity

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Impairments

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Topics covered



Formatting analog information	Sec. 2.4	pp. 62-75
Baseband modulation waveforms	Sec. 2.8	pp. 85-94
Sources of corruption	Sec. 2.5	pp. 76-79

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Sources of signal impairments

- Quantification noise
- Quantification Saturation
- Timing error or "jitter"
- Channel noise
 - ❑ thermal noise
 - ❑ interference
- Intersymbol interference (ISI)

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Sources of signal impairments

- Quantification noise
- Quantification Saturation
- Timing error or "jitter"
- Channel noise
 - ❑ thermal noise
 - ❑ interference
- Intersymbol interference (ISI)

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Intersymbol Interference

- Channel always limited in bandwidth
- Signal always limited in time
- ⇒ never limited in bandwidth
- Result of passing the signal through the channel:

pulse spreading

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Matlab Example

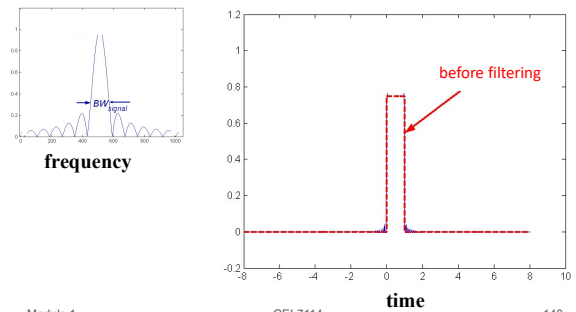
- A bandwidth limited **channel**
 - rectangular in frequency domain (ideal LPF)
 - easily defined bandwidth
- A time limited **signal**
 - rectangular in time domain
 - sinc in frequency domain
 - bandwidth defined by main lobe

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Source of ISI $BW_{channel}=10 \times BW_{signal}$

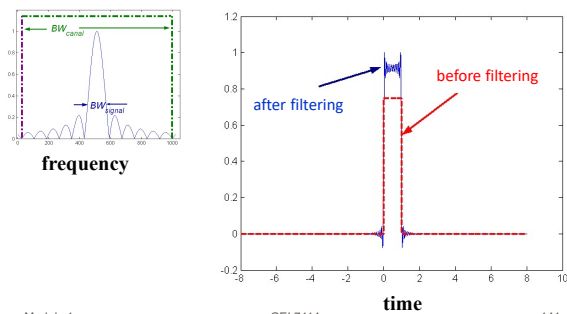


Module 1

GEL7114

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Source of ISI $BW_{channel}=10 \times BW_{signal}$

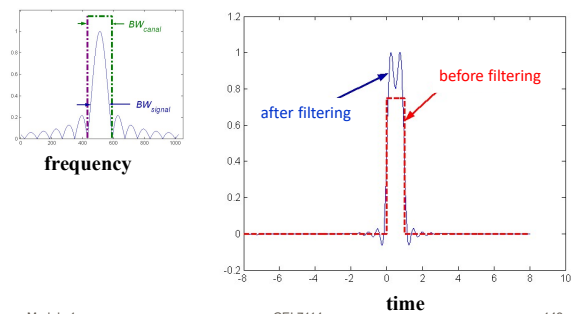


Module 1

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141

Source of ISI $BW_{channel}=BW_{signal}$

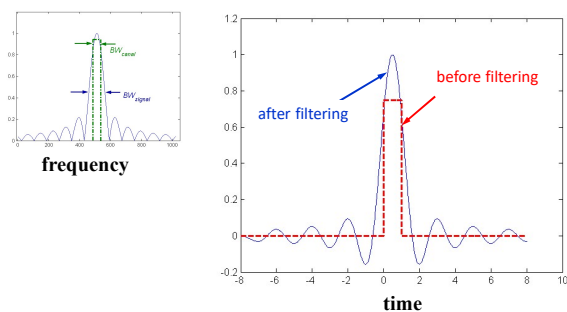


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Source of ISI $BW_{channel}=0.5 \times BW_{signal}$

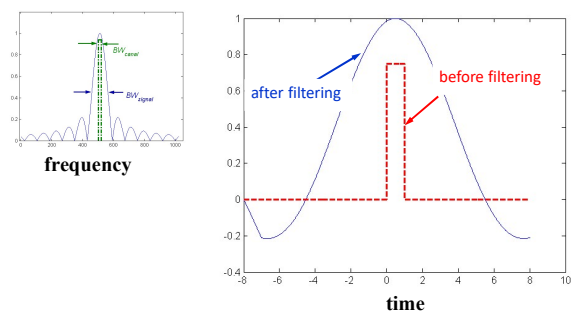


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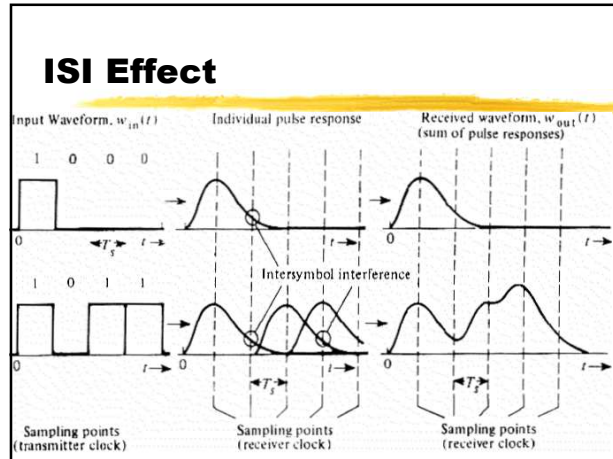
Source of ISI $BW_{channel}=0.1 \times BW_{signal}$



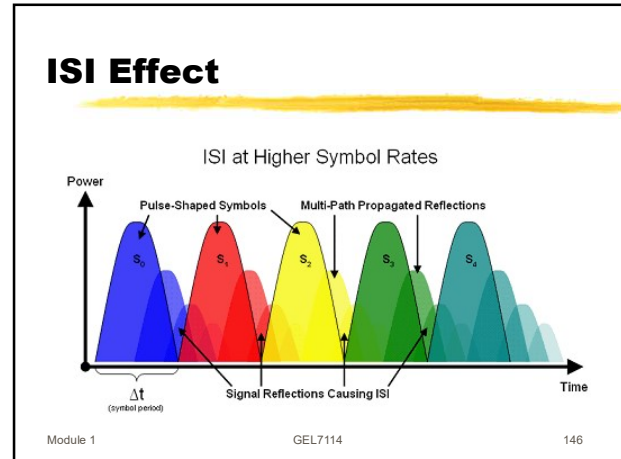
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