

Digital communications

Chapter 3 Linear sampling receiver

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Topics covered

- Linear sampling receiver (Sec. 3.1)
- Matched filter (Sec. 3.2.2)
- Signal space et Gram-Schmidt process (Sec. 3.1 and course notes)
- Error probability, MAP & ML receivers (Sec. 3.2, appendix B, course notes)
- OOK and antipodal Modulation (Sec. 3.2)
- Intersymbol interference (Sec. 3.3, 3.4.1, 3.4.2)
- Pages 104-151

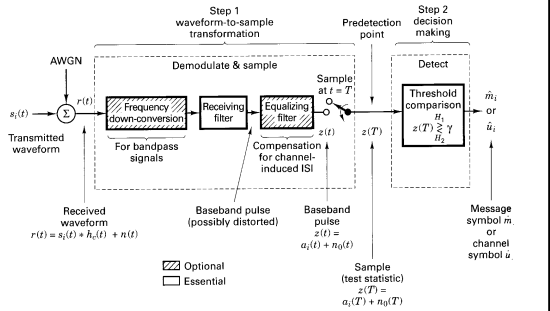
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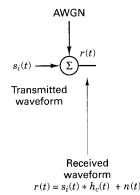
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Baseband



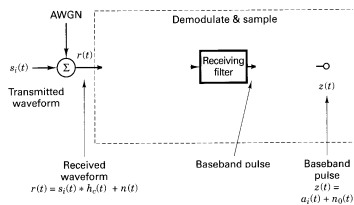
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Baseband



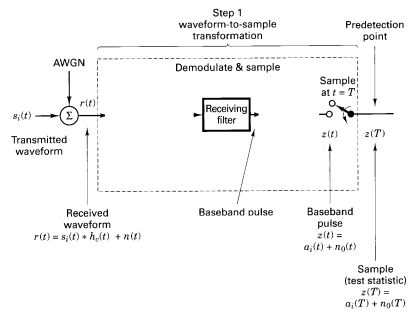
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Baseband



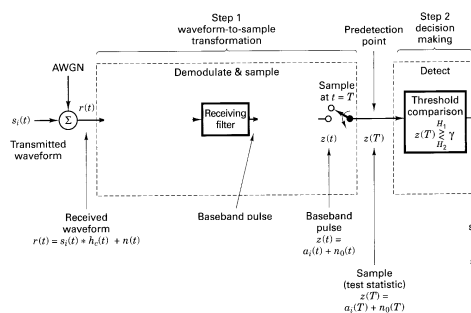
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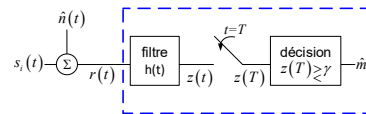
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Baseband



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Sampling receiver



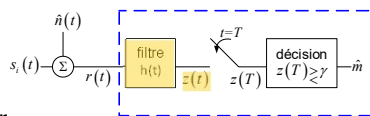
- Transmitted signal $s_i(t)$ or $s_0(t)$
- AWGN noise $\hat{n}(t)$
- Received signal $r(t) = s_i(t) + \hat{n}(t)$

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Filtering and sampling



- Filter
 - frequency response $H(f)$
 - impulse response $h(t)$
- Filter output
 - $z(t) = r(t) * h(t) = s_i(t) * h(t) + \hat{n}(t) * h(t)$
 - $= a_i(t) + n(t)$

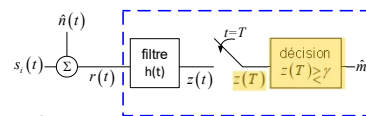
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Test statistic



- After sampling
 - $z(T) = a_i(T) + n(T)$
 - Test statistic
- Detection
 - comparison of the test statistic with a threshold γ
 - decision or estimate of the data $\hat{m}$

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Test statistic

- Part that comes from the signal
 - $a_0(T) = s_0(t) * h(t)|_{t=T}$
 - $a_i(T) = s_i(t) * h(t)|_{t=T}$
 - deterministic : $a_0(T)$ ou $a_i(T)$ (two numbers)
- Part that comes from the noise
 - $n(T) = \hat{n}(t) * h(t)|_{t=T}$
 - $\hat{n}(t)$ is a random Gaussian process
 - $\hat{n}(t) * h(t)$ is a random Gaussian process
 - $n(T)$ is a Gaussian random variable

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Noise

- Density of sampled noise
 - $p_n(n) = \frac{1}{\sqrt{2\pi}\sigma} e^{-n^2/2\sigma^2}$ $n \sim N(0, \sigma^2)$
 - Gaussian of zero mean and variance σ^2
- Density of the test statistic
 - $z(T) = a_i(T) + n(T)$ $z(T) \sim \eta(a_i(T), \sigma^2)$
 - The sum of a Gaussian random variable and a constant is a Gaussian random variable with a shifted mean.

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Vocabulary/notation

- Samples
 - n for $n(T)$
 - z for $z(T)$
 - a_i for $a_i(T)$
- Conditional density
 - $p_z(z|i \text{ sent})$ test statistic when the data "i" was sent
 - $p_z(z|0)$ et $p_z(z|1)$ for example

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Vocabulary/notation

- Conditional probability
 - $\Pr(s_i|z)$ is the probability that the data sent was "i" given the test statistic is z

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Conditional densities

- Density of $z(T)$ if "0" was sent

$$p_z(z|0) = p(z = a_0 + n) = p_n(n = z - a_0)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-a_0)^2}{2\sigma^2}}$$
- Density of $z(T)$ if "1" was sent

$$p_z(z|1) = p_z(z = a_1 + n) = p_n(n = z - a_1)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-a_1)^2}{2\sigma^2}}$$

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Chapter 3 Bayes Law

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Bayes Law

- Recall of this result in probability theory
- $p(B)p(A|B) = p(B|A)p(A)$
- In our application

$$p(s_i|z) = \frac{p(z|s_i)p(s_i)}{p(z)}$$

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Bayes Law

- "a priori" information
- $p(s_i)$ is the probability that s_i is sent
 - does not depend on the method of transmission or reception - it is a feature of the data
 - not always known

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"a posteriori" Probability

- The "a posteriori" probability
 - The probability $\text{Prob}(s_i|z)$ that the data "i" was sent given that we measured the test statistic z
 - Direct function of our receiver

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Vocabulary

$$p_z(z|i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-a_i)^2}{2\sigma^2}}$$

conditional probability
or LIKELIHOOD

a priori probability

$$p(s_i|z) = \frac{p(z|s_i)p(s_i)}{p(z)}$$

a posteriori probability

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MAP rule

- Maximizes "a posteriori" probability
- A receiver that chooses the data that maximizes probability after an observation is called a "MAP receiver"
 - Choose s_i such that $p(s_i|z)$ is maximized
 - $s_i \ni \max_j p(s_j|z) = p(s_i|z)$

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MAP Receiver

- Equivalent
 - Maximize $p(s_j|z) = \frac{p(z|s_j)p(s_j)}{p(z)}$
 - Maximize $p(z|s_j)p(s_j)$
- "a priori" probability
 - $p(s_j)$ side information that has nothing to do with the communication system
 - $p(s_j)$ not always known
 - weighting given to the Likelihood $p(z|s_j)$

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ML Receiver

- Maximize the Likelihood (maximum Likelihood)
 - Choose "i" such that $p(z|s_i)$ is maximized
 - $s_i \ni \max_j p(z|s_j) = p(z|s_i)$
- Equivalence between ML & MAP
 - for the case of equal "a priori" probabilities, $p(s_i) = \frac{1}{M}$ the ML and MAP receivers are identical

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Chapter 3 Binary ML

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ML binary example

➤ M-ary equation

$$\arg \max_j p(z|s_j) = \arg \max_j e^{-(z-a_j)^2/2\sigma^2}$$

➤ Binary case

➤ Choose 1 if

$$e^{-(z-a_1)^2/2\sigma^2} > e^{-(z-a_0)^2/2\sigma^2}$$

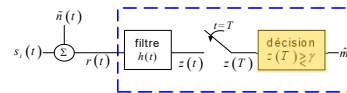
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Filtering and sampling



➤ Sortie After sampling

$$z(T) = a_i(T) + n(T)$$

signal $\rightarrow$ $a_i(T)$ $\rightarrow$ AWGN noise $\rightarrow$ $n(T)$

$$e^{-(z-a_1)^2/2\sigma^2} > e^{-(z-a_0)^2/2\sigma^2}$$

règle ML

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Choose 1 if ...

ML rule, binary case

$$\text{➤ } e^{-(z-a_1)^2/2\sigma^2} > e^{-(z-a_0)^2/2\sigma^2}$$

$$\text{➤ } -(z-a_1)^2/2\sigma^2 > -(z-a_0)^2/2\sigma^2$$

$$\text{➤ } (z-a_1)^2 < (z-a_0)^2$$

$$\text{➤ } z \text{ closer to } a_1$$

$$\text{➤ } z > \frac{a_0 + a_1}{2} \text{ assuming } a_1 > a_0$$

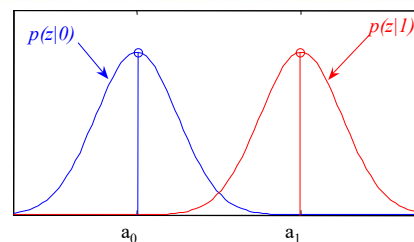
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Likelihood



$$z(t) = s_i(t) * h_c(t) * h(t) + \hat{n}(t) * h(t)$$

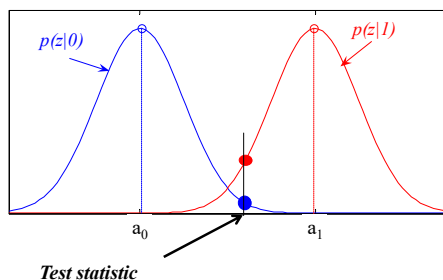
$$= a_i(t) + n(t)$$

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Likelihood



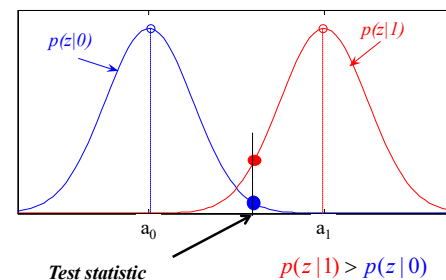
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Likelihood



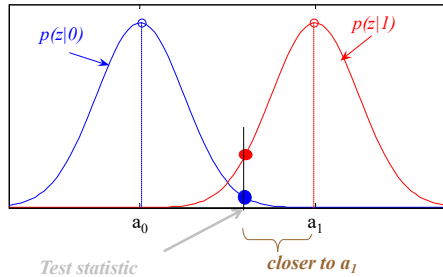
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Likelihood



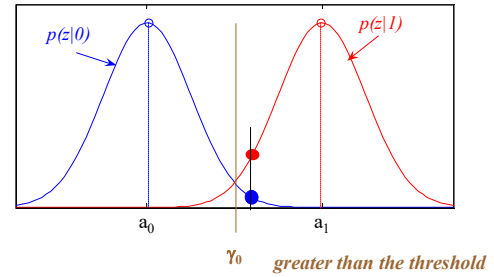
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Likelihood



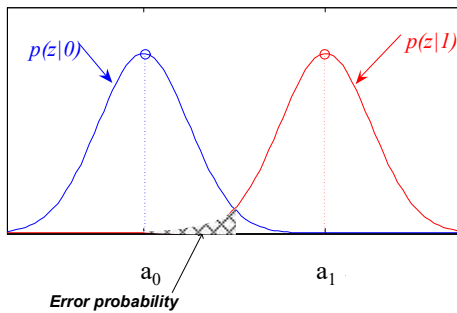
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Likelihood



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Chapter 3 Binary MAP

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MAP binary example

➤ M-ary equation

$$\arg \max_j p(z|s_j)p(s_j) = \arg \max_j p(s_j)e^{-(z-a_j)^2/2\sigma^2}$$

➤ Binary case

Choose 1 if

$$p(s_1)e^{-(z-a_1)^2/2\sigma^2} > p(s_0)e^{-(z-a_0)^2/2\sigma^2}$$

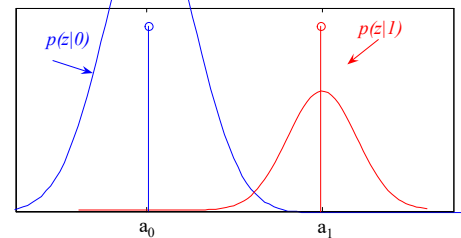
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Likelihood for MAP case

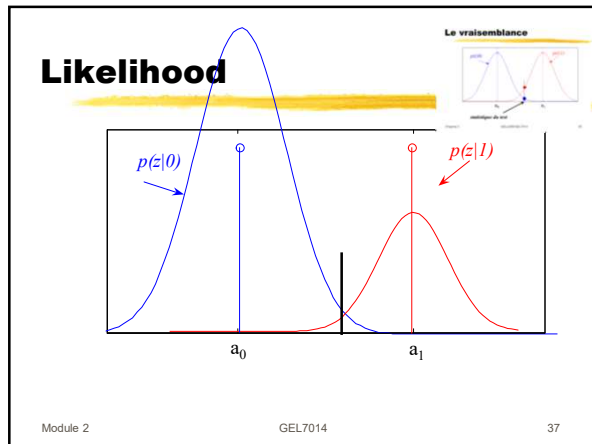


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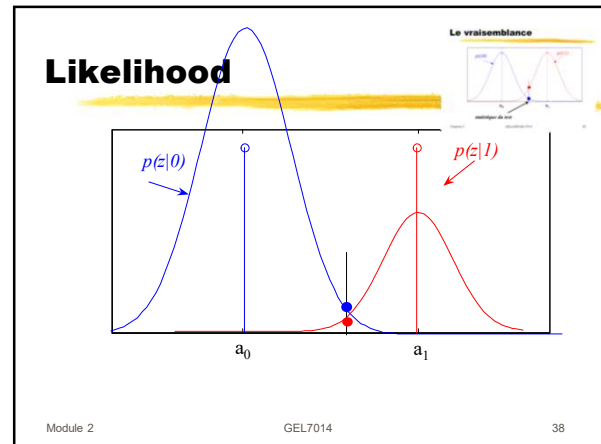
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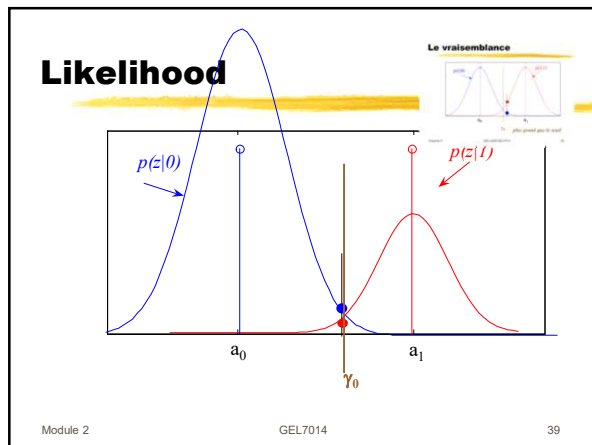
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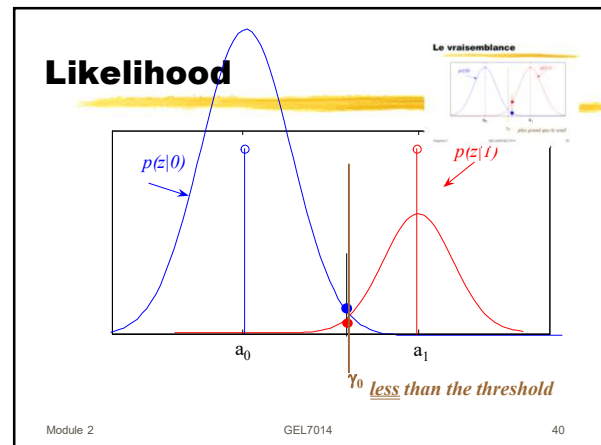
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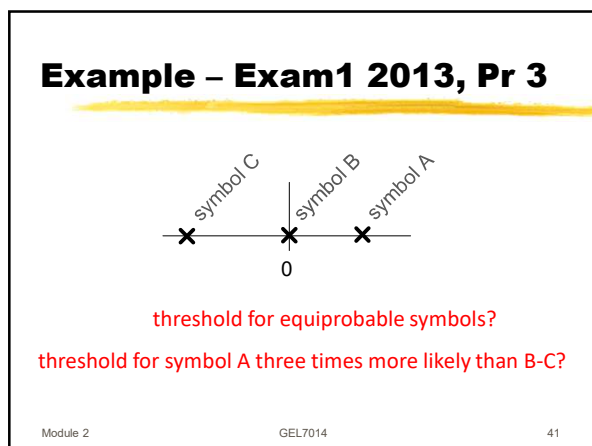
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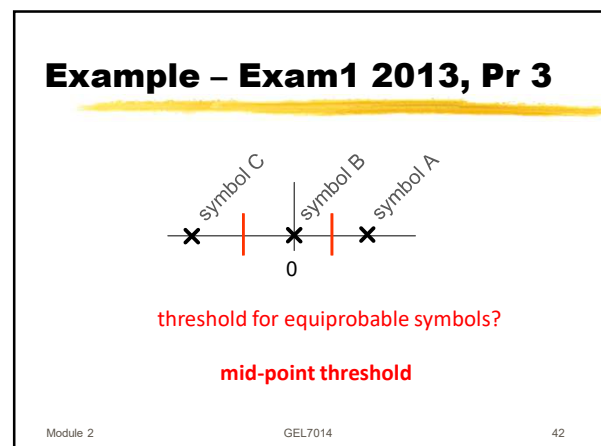
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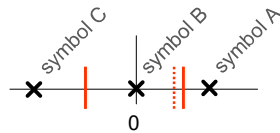


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Example – Exam1 2013, Pr 3



threshold for symbol A three times more likely than B-C?

a shifted threshold

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Summary

- Sampling receivers
 - Signal and AWGN noise
 - $z(T)$ Test statistic (value at the time of sampling)
 - Decision of the transmitted symbol based only on $z(T)$

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Summary

- MAP and ML receivers
 - Definitions
 - Binary case
- Choosing the signal and the filter
 - $s(t)$ and $h(t)$ still arbitrary

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Matched filter

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Topics covered

- Linear sampling receiver (Sec. 3.1)
- Matched filter (Sec. 3.2.2)
- Signal space et Gram-Schmidt process (Sec. 3.1 and course notes)
- Error probability & receivers MAP & ML (Sec. 3.2, appendix B, course notes)
- Modulation OOK et antipodal (Sec. 3.2)
- Intersymbol interference (Sec. 3.3, 3.4.1, 3.4.2)
- Pages 104-151

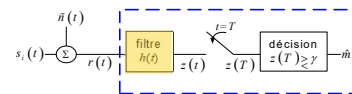
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Filtering and sampling



- Filter
 - frequency response $H(f)$
- Output after sampling
 - $z(T) = a_i(T) + n(T)$

signal

noise

How to choose $H(f)$?

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Instantaneous power

➤ Signal

$$a_i(T) = s(t) * h(t) \Big|_{t=T} = \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df = \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df$$

□ Instantaneous power

$$a_i^2(T) = \left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2$$

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PSD at filter output

➤ LTIS with frequency response $H(f)$

➤ input PSD $G_X(f)$

➤ output PSD

$$G_Y(f) = G_X(f) |H(f)|^2$$

$$G_X(f) = \frac{N_0}{2}$$

$$G_Y(f) = \frac{N_0}{2} |H(f)|^2$$

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Average power

➤ Noise

□ Input power spectral density

$$G_X(f) = \frac{N_0}{2}$$

□ Power spectral density after filtering

$$G_Y(f) = \frac{N_0}{2} |H(f)|^2$$

□ Average power

$$\int_{-\infty}^{\infty} G_Y(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$$

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Signal-to-noise ratio

➤ We want to choose $H(f)$ to maximize

$$\frac{\left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

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Schwartz inequality

$$\left| \int_{-\infty}^{\infty} g_1(x) g_2(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |g_1(x)|^2 dx \cdot \int_{-\infty}^{\infty} |g_2(x)|^2 dx$$

➤ Equality for

$$g_1(x) = k g_2^*(x)$$

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Signal-to-noise ratio

$$\frac{\left| \int_{-\infty}^{\infty} g_1(x) g_2(x) dx \right|^2}{\int_{-\infty}^{\infty} |g_2(x)|^2 dx} \leq \frac{\int_{-\infty}^{\infty} |g_1(x)|^2 dx}{\int_{-\infty}^{\infty} |g_2(x)|^2 dx} \quad \frac{\left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

➤ Define

$$g_1 = H(f) \quad g_2 = S(f) e^{-j2\pi fT}$$

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Inequality

$$\left| \int_{-\infty}^{\infty} g_1(x) g_2(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |g_1(x)|^2 dx \cdot \int_{-\infty}^{\infty} |g_2(x)|^2 dx = \left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2 \cdot \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$$

➤ Define

$$g_1 = H(f) \quad g_2 = S(f) e^{j2\pi fT}$$

$$= \int_{-\infty}^{\infty} |S(f)|^2 df$$

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Inequality

$$\left| \int_{-\infty}^{\infty} g_1(x) g_2(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |g_1(x)|^2 dx \cdot \int_{-\infty}^{\infty} |g_2(x)|^2 dx$$

$$\left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \cdot \int_{-\infty}^{\infty} |S(f)|^2 df$$

$$\left| \int_{-\infty}^{\infty} g_1(x) g_2(x) dx \right|^2 = \left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2 \leq$$

$$\int_{-\infty}^{\infty} |g_1(x)|^2 dx = \int_{-\infty}^{\infty} |H(f)|^2 df$$

$$\int_{-\infty}^{\infty} |g_2(x)|^2 dx = \int_{-\infty}^{\infty} |S(f) e^{j2\pi fT}|^2 df = \int_{-\infty}^{\infty} |S(f)|^2 df$$

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Signal-to-noise ratio

$$\frac{\left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \leq \frac{\int_{-\infty}^{\infty} |S(f) H(f) e^{j2\pi fT} df|^2}{\int_{-\infty}^{\infty} |H(f)|^2 df} \leq \int_{-\infty}^{\infty} |S(f)|^2 df$$

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Signal-to-noise ratio

$$\frac{\left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \leq \frac{\int_{-\infty}^{\infty} |S(f) H(f) e^{j2\pi fT} df|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |S(f)|^2 df$$

signal energy

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continuing...

Inégalité de Schwartz

$$\left| \int_{-\infty}^{\infty} g_1(x) g_2(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |g_1(x)|^2 dx \int_{-\infty}^{\infty} |g_2(x)|^2 dx$$

Égalité pour

$$g_1(x) = k g_2^*(x)$$

$$\frac{\left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |S(f)|^2 df$$

➤ Égalité for

$$g_1 = k g_2^*$$

$$H(f) = k S^*(f) e^{j2\pi fT}$$

Matched filter

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Matched filter

➤ Equality for $H(f) = k S^*(f) e^{j2\pi fT}$

$$h(t) = TF^{-1} \left\{ k S^*(f) e^{j2\pi fT} \right\} = \begin{cases} k s(T-t) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

assuming $s(t) = 0$ for t other than $0 \leq t \leq T$

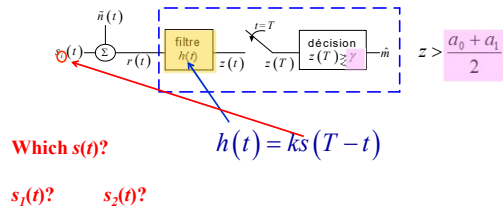
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Filtering and sampling



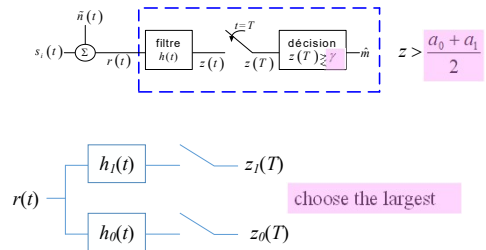
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Filtering and sampling



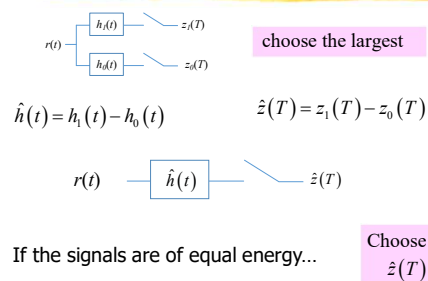
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Filtering and sampling



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$$Q_0(\tau) < 0$$

$$Q_1(\tau) > 0$$

$$Q_0(\tau) = \int_0^{\tau} s_0(t) * [s_1(t) - s_0(t)] dt$$

$$= \int_0^{\tau} s_0(t) * s_1(t) dt - \int_0^{\tau} s_0(t) * s_0(t) dt$$

$$Q_1(\tau) = \int_0^{\tau} s_1(t) * [s_1(t) - s_0(t)] dt$$

$$= \int_0^{\tau} s_1(t) * s_1(t) dt - \int_0^{\tau} s_1(t) * s_0(t) dt$$

$$\frac{Q_0(\tau) + Q_1(\tau)}{2} = \frac{\int_0^{\tau} s_0(t) * s_1(t) dt - \int_0^{\tau} s_0(t) * s_0(t) dt + \int_0^{\tau} s_1(t) * s_1(t) dt - \int_0^{\tau} s_1(t) * s_0(t) dt}{2}$$

$$= \frac{s_1(t) * s_1(t) - s_0(t) * s_0(t)}{2}$$

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Signal space

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- Pages 104-151

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Conclusions to date

- Maximize a posteriori probability (MAP)
- Maximize the Likelihood (ML)
- Use a matched filter to maximize the SNR
- The choice of waveform is arbitrary
- We only considered 2 symbols...
- Next: generalization M symbols & BER

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Signal space

- Motivation
 - Mathematics to attack ML & MAP
 - Decision rules
 - Geometric intuition
 - Error probability calculation
- Vector space
 - M symbols can be represented in a finite vector space (at most M dimensional)
 - Noise has a space of ∞ dimension

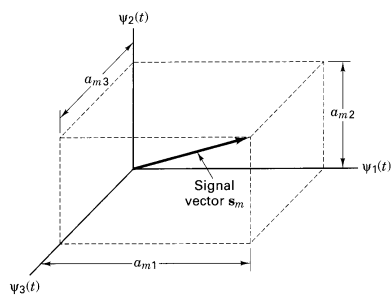
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Vector space



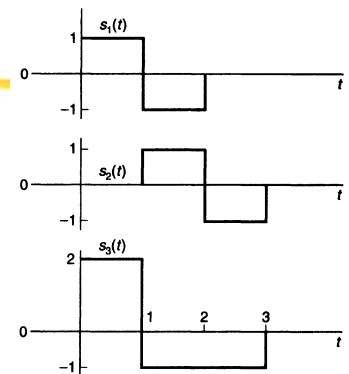
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Example



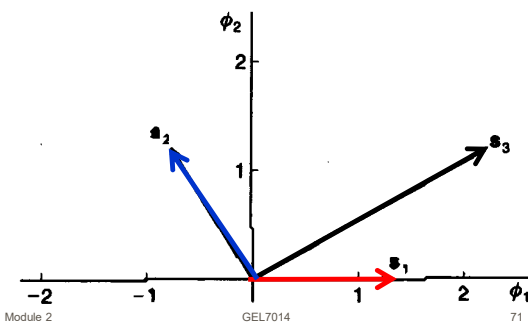
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Solution



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Orthonormal basis

- Basis vectors

$$\{\psi_1(t), \psi_2(t), \dots, \psi_N(t)\}$$

$$\langle \psi_i(t), \psi_j(t) \rangle = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$
- Inner product

$$\langle \psi_i(t), \psi_j(t) \rangle \triangleq \int_0^T \psi_i(t) \psi_j(t) dt$$

$$M \text{ symbols} \Rightarrow N \leq M$$

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Signal vector

Signal in vector form

- $s_i(t) = \sum_{j=1}^N a_{ij} \psi_j(t)$
- a_{ij} are the coefficients of $s_i(t)$ in signal space
- $\underline{s}_i = [a_{i1} \ a_{i2} \ \dots \ a_{iN}]$
- $a_{ik} = \langle s_i(t), \psi_k(t) \rangle$
- $\|s_i(t)\|^2 = \sum_{j=1}^N a_{ij}^2$

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Signal vector

Signal in vector form

- $s_i(t) = \sum_{j=1}^N a_{ij} \psi_j(t)$
 - a_{ij} are the coefficients of $s_i(t)$ in signal space
 - $\underline{s}_i = [a_{i1} \ a_{i2} \ \dots \ a_{iN}]$
 - $a_{ik} = \langle s_i(t), \psi_k(t) \rangle$
 - $\|s_i(t)\|^2 = \sum_{j=1}^N a_{ij}^2$
- Handwritten notes:*
 $\langle \sum_{j=1}^N a_{ij} \psi_j(t), \psi_k(t) \rangle$
 $= \sum_{j=1}^N a_{ij} \langle \psi_j(t), \psi_k(t) \rangle$
 $= a_{ik}$

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Orthonormal basis

Norm

$$\langle s_i(t), s_i(t) \rangle = \int_0^T s_i^2(t) dt = \|s_i(t)\|^2$$

Signal energy

Average signal energy for a constellation of M symbols

$$E_s = \frac{1}{M} \sum_{i=1}^M \|s_i(t)\|^2$$

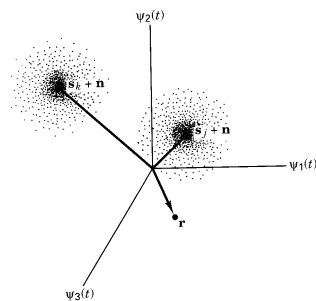
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Noise - vector space



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Basis extension

Noise can fall into a larger space than the signal space

Infinite dimension

Form a basis with the same first N vectors

$$\{\psi_1(t), \psi_2(t), \dots, \psi_N(t), \psi_{N+1}(t), \dots\}$$

$$\underline{n} = [n_1 \ n_2 \ \dots \ n_N \ n_{N+1} \ \dots] \quad n_i = \langle n(t), \psi_i(t) \rangle$$

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Basis extension

Symbol

Symbol

Only non-zero coefficients are in the signal space

$$\underline{s}_i = [a_{i1} \ a_{i2} \ \dots \ a_{iN} \ 0 \ 0 \ \dots]$$

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Received signal

➤ $r(t) = s_i(t) + n(t)$

➤ vector form

$$\underline{r} = \underline{s}_i + \underline{n}$$

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Digital communications

Chapter 3 Gram Schmidt Process

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Gram Schmidt Process

➤ Generate an orthonormal basis from M signals $\{s_i(t)\}_{i=1}^M$

➤

1- $\psi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$

where $E_1 = \int_0^T s_1^2(t) dt$

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Gram Schmidt Stage 2

2- $\theta_2(t) = s_2(t) - \langle s_2(t), \psi_1(t) \rangle \psi_1(t)$

we remove the part of the second signal that falls into the signal space of $s_1(t)$
[i.e. the projection of $s_2(t)$ on $s_1(t)$]

$$E_2 = \int_0^T \theta_2^2(t) dt$$

$$\psi_2(t) = \frac{\theta_2(t)}{\sqrt{E_2}}$$

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Gram Schmidt i^{th} step

i- $\theta_i(t) = s_i(t) - \sum_{k=1}^{i-1} \langle s_i(t), \psi_k(t) \rangle \psi_k(t)$

we remove the part of $s_i(t)$ which can be represented by the basic vectors already found

$$E_i = \int_0^T \theta_i^2(t) dt \quad \psi_i(t) = \frac{\theta_i(t)}{\sqrt{E_i}}$$

➤ When $\psi_i(t) = 0$ there is no other basis vector at this stage

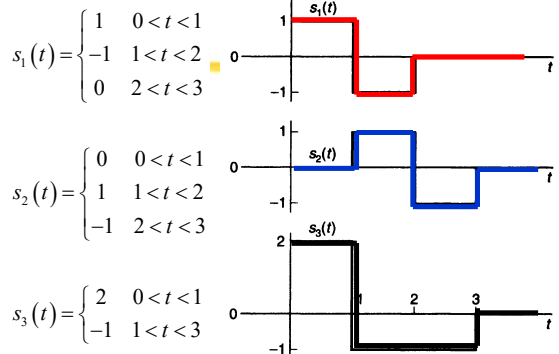
➤ The process stops at the latest when $i=M$

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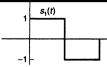
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Step 1

$$s_1(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$


$$\psi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$E_1 = \int_0^3 s_1^2(t) dt = \int_0^1 (1)^2 dt + \int_1^2 (-1)^2 dt = 2$$

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{2}} = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

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Step 2a

$$s_2(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ -1 & 2 < t < 3 \end{cases} \quad \psi_1(t) = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

$$\theta_2(t) = s_2(t) - \langle s_2(t), \psi_1(t) \rangle \psi_1(t)$$

$$\begin{aligned} \langle s_2(t), \psi_1(t) \rangle &= \int_0^3 s_2(t) \psi_1(t) dt \\ &= \int_0^1 0 \cdot \frac{1}{\sqrt{2}} dt + \int_1^2 1 \cdot \frac{-1}{\sqrt{2}} dt + \int_2^3 -1 \cdot 0 dt \\ &= \frac{-1}{\sqrt{2}} \end{aligned}$$

$$\theta_2(t) = s_2(t) - \frac{-1}{\sqrt{2}} \psi_1(t)$$

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Step 2b

$$\theta_2(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ -1 & 2 < t < 3 \end{cases} - \begin{cases} 1/2 & 0 < t < 1 \\ -1/2 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

$$= \begin{cases} 1/2 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ -1 & 2 < t < 3 \end{cases}$$

$$E_2 = \int_0^3 \theta_2^2(t) dt = \int_0^1 \left(\frac{1}{2}\right)^2 dt + \int_1^2 (-1)^2 dt + \int_2^3 (-1)^2 dt = \frac{2}{4} + 1 = \frac{3}{2}$$

$$\psi_2(t) = \theta_2(t) \cdot \sqrt{\frac{2}{3}} = \begin{cases} 1/\sqrt{6} & 0 < t < 1 \\ -2/\sqrt{6} & 1 < t < 2 \\ -2/\sqrt{6} & 2 < t < 3 \end{cases}$$

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Step 3a

$$\psi_1(t) = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases} \quad \psi_2(t) = \begin{cases} 1/\sqrt{6} & 0 < t < 1 \\ -2/\sqrt{6} & 1 < t < 2 \\ -2/\sqrt{6} & 2 < t < 3 \end{cases}$$

$$\theta_3(t) = s_3(t) - \langle s_3(t), \psi_1(t) \rangle \psi_1(t) - \langle s_3(t), \psi_2(t) \rangle \psi_2(t)$$

$$\begin{aligned} \langle s_3(t), \psi_1(t) \rangle &= \frac{1}{\sqrt{2}} \int_0^1 2 \cdot 1 dt + \frac{1}{\sqrt{2}} \int_1^2 (-1) \cdot (-1) dt + \int_2^3 (-1) \cdot 0 dt \\ &= \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \langle s_3(t), \psi_2(t) \rangle &= \int_0^1 2 \cdot \frac{1}{\sqrt{6}} dt + \int_1^2 (-1) \cdot \frac{-2}{\sqrt{6}} dt + \int_2^3 (-1) \cdot \left(-\frac{2}{\sqrt{6}}\right) dt \\ &= \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{5}{\sqrt{6}} \end{aligned}$$

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Step 3b

$$s_3(t) \quad \langle s_3(t), \psi_1(t) \rangle \psi_1(t) \quad \langle s_3(t), \psi_2(t) \rangle \psi_2(t)$$

$$\begin{aligned} \theta_3(t) &= \begin{cases} 2 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ -1 & 2 < t < 3 \end{cases} - \frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases} - \frac{5}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{cases} 1 & 0 < t < 1 \\ -2 & 1 < t < 2 \\ -2 & 2 < t < 3 \end{cases} \\ &= \begin{cases} 2 - 3/2 - 5/6 & 0 < t < 1 \\ -1 + 3/2 - 10/6 & 1 < t < 2 \\ -1 - 0 + 10/6 & 2 < t < 3 \end{cases} \\ &= \begin{cases} -1/6 & 0 < t < 1 \\ -1/6 & 1 < t < 2 \\ 1/3 & 2 < t < 3 \end{cases} \end{aligned}$$

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Result

➤ Two dimensional basis

$$\psi_1(t) = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases} \quad \psi_2(t) = \begin{cases} 1/\sqrt{6} & 0 < t < 1 \\ -2/\sqrt{6} & 1 < t < 2 \\ -2/\sqrt{6} & 2 < t < 3 \end{cases}$$

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Result

$\psi_1(t) = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$
 $\psi_2(t) = \begin{cases} 1/\sqrt{6} & 0 < t < 2 \\ -2/\sqrt{6} & 2 < t < 3 \end{cases}$

$\langle s_1(t), \psi_1(t) \rangle = \frac{E_1}{\sqrt{E_1}} = \sqrt{E_1} = \sqrt{2}$
 $\langle s_1(t), \psi_2(t) \rangle = 0$ by construction
 $\langle s_2(t), \psi_1(t) \rangle = -1/\sqrt{2}$ see step 2a
 $\langle s_2(t), \psi_2(t) \rangle = \int_1^2 (1) \left(\frac{1}{\sqrt{6}} \right) dt + \int_2^3 (-1) \left(-\frac{2}{\sqrt{6}} \right) dt = 3/\sqrt{6}$
 $\langle s_3(t), \psi_1(t) \rangle = 3/\sqrt{2}$ see step 3a
 $\langle s_3(t), \psi_2(t) \rangle = 3/\sqrt{6}$

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Solution

$\underline{s}_1 = [\sqrt{2} \ 0]$
 $\underline{s}_2 = [-\frac{1}{\sqrt{2}} \ \frac{3}{\sqrt{6}}]$
 $\underline{s}_3 = [\frac{3}{\sqrt{2}} \ \frac{3}{\sqrt{6}}]$

$\|\underline{s}_1\| = \sqrt{2}$
 $\|\underline{s}_2\| = \sqrt{\frac{1}{2} + \frac{9}{6}} = \sqrt{2}$
 $\|\underline{s}_3\| = \sqrt{\frac{9}{2} + \frac{9}{6}} = \sqrt{6}$

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Signal space analysis

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Topics covered

- Linear sampling receiver (Sec. 3.1)
- Matched filter (Sec. 3.2.2)
- Signal space et Gram-Schmidt process (Sec. 3.1 and course notes)
- Error probability & receivers MAP & ML (Sec. 3.2, appendix B, course notes)
- Modulation OOK et antipodal (Sec. 3.2)
- Intersymbol interference (Sec. 3.3, 3.4.1, 3.4.2)
- Pages 104-151

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Signal space analysis

- Received signal
 - Time domain

$$r(t) = s(t) + n(t)$$
 - Vector form

$$\underline{r} = \underline{s} + \underline{n}$$
- How to choose a transmitted signal from an observation of $\underline{r}$
- MAP Maximize "a posteriori" probability
 - ML Maximize the Likelihood
- **We need to analyze the noise ...**

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Signal space

Vocabulaire/notation

- Échantillons
 - x pour $x(t)$
 - $\underline{x}$ pour $\underline{x}(t)$
 - $\underline{x}$ pour $\underline{x}(t)$
- Densité conditionnelle
 - $p(\cdot | \cdot)$ densité de la statistique du test quand le symbole "1" a été envoyé
 - $p(\cdot | \cdot)$ et $p(\cdot | \cdot)$ par exemple

$$r(t) = \sum_{j=1}^{\infty} r_j \psi_j(t) \quad \underline{r} = [r_1 \ r_2 \ \dots \ r_n \ r_{n+1} \ \dots]$$

$$n(t) = \sum_{j=1}^{\infty} n_j \psi_j(t) \quad \underline{n} = [n_1 \ n_2 \ \dots \ n_n \ n_{n+1} \ \dots]$$

$$s_i(t) = \sum_{j=1}^{\infty} s_{ij} \psi_j(t) \quad \underline{s}_i = [s_{i,1} \ s_{i,2} \ \dots \ s_{i,n} \ 0 \ 0 \ \dots]$$

$$\Pr(r(t) | s_i(t)) = \Pr(r(t) = s_i(t) + n(t))$$

$$= \Pr(\underline{r} = \underline{s}_i + \underline{n})$$

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Noise in signal space

➤ Noise vector

$$\underline{n} = [n_1 \ n_2 \ \dots] \quad n_j = \int_0^T n(t) \psi_j(t) dt$$

➤ Recall $n(t)$ Gaussian process

- $En(t) = 0$
- $\text{PSD}[n(t)] = G_N(f) = \frac{N_0}{2} \quad \forall f \quad R_N(\tau) = \frac{N_0}{2} \delta(\tau)$
- $En(t)n(t+\tau) = 0 \quad \tau \neq 0$

➤ $n(t)$ Gaussian

$\Rightarrow n_j$ Gaussian (integration is a linear operation)

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Noise analysis

➤ Expected value

$$\begin{aligned} En_j &= E \int_0^T n(t) \psi_j(t) dt \\ &= \int_0^T E \{n(t)\} \psi_j(t) dt \\ &= \int_0^T 0 \psi_j(t) dt = 0 \end{aligned}$$

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Noise analysis

➤ Variance

$$\begin{aligned} En_j^2 &= E \int_0^T n(t) \psi_j(t) dt \cdot \int_0^T n(z) \psi_j(z) dz \\ &= E \int_0^T \int_0^T n(t) n(z) \psi_j(t) \psi_j(z) dz dt \\ &= \int_0^T \int_0^T E \{n(t) n(z)\} \psi_j(t) \psi_j(z) dz dt \\ &= \int_0^T \int_0^T \frac{N_0}{2} \delta(t-z) \psi_j(t) \psi_j(z) dz dt \\ &= \frac{N_0}{2} \int_0^T \psi_j^2(t) dt = \frac{N_0}{2} \end{aligned}$$

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Independence of coefficients

➤ Covariance, i.e., suppose $i \neq j$

$$\begin{aligned} En_i n_j &= E \int_0^T n(t) \psi_i(t) dt \int_0^T n(z) \psi_j(z) dz \\ &= \int_0^T \int_0^T E \{n(t) n(z)\} \psi_i(t) \psi_j(z) dz dt \\ &= \int_0^T \int_0^T \frac{N_0}{2} \delta(t-z) \psi_i(t) \psi_j(z) dz dt \\ &= \frac{N_0}{2} \int_0^T \psi_i(t) \psi_j(t) dt \\ &= \frac{N_0}{2} \langle \psi_i(t) \psi_j(t) \rangle = 0 \end{aligned}$$

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Independence of coefficients

➤ The coefficients are not correlated for white noise in an orthonormal base

$$\text{➤} \quad En_i n_j = \frac{N_0}{2} \langle \psi_i(t) \psi_j(t) \rangle = 0$$

➤ For Gaussian random variables, uncorrelated $\Rightarrow$ Independent

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Noise Summary

➤ Gaussian of zero mean

$$\text{➤} \quad n_j \sim N\left(0, \frac{N_0}{2}\right) \quad \sigma^2 = \frac{N_0}{2}$$

➤ Independent $En_i n_j = 0$ for $j \neq i$

➤ Vector density = joint density of coefficients

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Noise density

- By independence

$$P_{\underline{n}}(\underline{n}) = \prod_{j=1}^{\infty} P_{n_j}(n_j) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-n_j^2/2\sigma^2}$$

$$\begin{aligned} p(r(t)|s_i(t)) &= p(r(t) = s_i(t) + n(t)) \\ &= p(\underline{r} = \underline{s}_i + \underline{n}) \\ &= p(\underline{n} = \underline{r} - \underline{s}_i) \\ &= \prod_{j=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(r_j - s_{j,i})^2/2\sigma^2} \end{aligned}$$

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Detectors

- ML (Likelihood)

Maximize the likelihood, i.e. choosing the $s_i(t)$ which maximizes the likelihood of receiving $r(t)$

$$s_i(t) \text{ such that } \arg \max_j P(\underline{r}|\underline{s}_j) = i$$

- MAP (*a posteriori*)

Choose transmitted signal to maximize the "a posteriori" probability

$$s_i \text{ such that } \arg \max_j P(\underline{r}|\underline{s}_j) P(\underline{s}_j) = i$$

a priori probability

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MAP choice

$$\begin{aligned} \arg \max_j P(\underline{r}|\underline{s}_j) P(\underline{s}_j) &= \arg \max_j P(\underline{r} = \underline{s}_j + \underline{n}) P(\underline{s}_j) \\ &= \arg \max_j P(\underline{n} = \underline{r} - \underline{s}_j) P(\underline{s}_j) \\ &= \arg \max_j \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(r_k - s_{j,k})^2/2\sigma^2} P(\underline{s}_j) \\ &= \arg \max_j \prod_{k=1}^{\infty} e^{-(r_k - s_{j,k})^2/2\sigma^2} P(\underline{s}_j) \\ &= \arg \max_j e^{-\frac{1}{2\sigma^2} \sum_{k=1}^{\infty} (r_k - s_{j,k})^2} P(\underline{s}_j) \end{aligned}$$

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MAP choice

$$\begin{aligned} \arg \max_j P(\underline{r}|\underline{s}_j) P(\underline{s}_j) &= \arg \max_j e^{-\frac{1}{2\sigma^2} \sum_{k=1}^{\infty} (r_k - s_{j,k})^2} P(\underline{s}_j) \\ &= \arg \max_j \ln \left[e^{-\frac{1}{2\sigma^2} \sum_{k=1}^{\infty} (r_k - s_{j,k})^2} P(\underline{s}_j) \right] \\ &= \arg \max_j \left[-\frac{1}{2\sigma^2} \sum_{k=1}^{\infty} (r_k - s_{j,k})^2 + \ln P(\underline{s}_j) \right] \\ &= \arg \max_j \left[\sum_{k=1}^{\infty} (r_k - s_{j,k})^2 + 2\sigma^2 \ln P(\underline{s}_j) \right] \\ &= \arg \min_j \left[\sum_{k=1}^{\infty} (r_k - s_{j,k})^2 - 2\sigma^2 \ln P(\underline{s}_j) \right] \end{aligned}$$

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MAP choice

- We want $\arg \min_j \left[\sum_{k=1}^{\infty} (r_k - s_{j,k})^2 - 2\sigma^2 \ln P(\underline{s}_j) \right]$

- Theorem of irrelevance

$$s_{j,k} = 0 \quad \forall \quad k > N$$

Elements for $k > N$ in the sum do not change the minimization

norm in the finite basis of the signal space

$$\begin{aligned} \arg \min_j \left[\sum_{k=1}^{\infty} (r_k - s_{j,k})^2 - 2\sigma^2 \ln P(\underline{s}_j) \right] \\ &= \arg \min_j \left[\sum_{k=1}^N (r_k - s_{j,k})^2 - 2\sigma^2 \ln P(\underline{s}_j) \right] \\ &= \arg \min_j \left[\|\underline{r} - \underline{s}_j\|^2 - 2\sigma^2 \ln P(\underline{s}_j) \right] \end{aligned}$$

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MAP choice

Choose the signal that is closer to the received signal modulo a weighting for a priori probability

$$\arg \min_j \left[\|\underline{r} - \underline{s}_j\|^2 - 2\sigma^2 \ln P(\underline{s}_j) \right]$$

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MAP choice

Choose the signal
received signal most
likely a priori probability

$$\text{Arg} \min_j \left[\|r - s_j\|^2 - 2\sigma^2 \ln P(s_j) \right]$$

Choisir 1 si ... règle ML, cas binaire

- $e^{-(z-a_1)^2/2\sigma^2} > e^{-(z-a_2)^2/2\sigma^2}$
- $-(z-a_1)^2/2\sigma^2 > -(z-a_2)^2/2\sigma^2$
- $(z-a_1)^2 < (z-a_2)^2$
- z plus proche de a_1
- $z > \frac{a_2 + a_1}{2}$ en supposant $a_1 > a_2$

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ML choice

Choose the signal closest to the received
signal

$$\text{Arg} \min_j \left[\|r - s_j\|^2 \right]$$

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ML choice

Choose the signal
signal

$$\text{Arg} \min_j \left[\|r - s_j\|^2 \right]$$

Choose 1 if ... ML rule, binary case

- $e^{-(z-a_1)^2/2\sigma^2} > e^{-(z-a_2)^2/2\sigma^2}$
- $-(z-a_1)^2/2\sigma^2 > -(z-a_2)^2/2\sigma^2$
- $(z-a_1)^2 < (z-a_2)^2$
- z closer to a_1
- $z > \frac{a_2 + a_1}{2}$ assuming $a_1 > a_2$

signal space analysis, and
linear sampling receiver probability
analysis
* give the same results *

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ML choice

Signal space analysis will facilitate extension to
multidimensional cases

$$\text{Arg} \min_j \left[\|r - s_j\|^2 \right]$$

signal space analysis, and
linear sampling receiver probability
analysis
* give the same results *

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Example: Exam1, 2014, Pr 2

$$\begin{aligned} s_1(t) &= 1, & 0 \leq t \leq 1 \\ s_2(t) &= \cos 2\pi t, & 0 \leq t \leq 1 \\ s_3(t) &= \cos^2 \pi t, & 0 \leq t \leq 1 \end{aligned}$$

Do these signals have the equal energy? What is the
average energy per symbol?
Give an orthogonal basis for these three signals.

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Example: Exam1, 2014, Pr 2

$$\begin{aligned} s_1(t) &= 1, & 0 \leq t \leq 1 & \quad \mathbf{1} \\ s_2(t) &= \cos 2\pi t, & 0 \leq t \leq 1 & \quad \mathbf{1/2} \\ s_3(t) &= \cos^2 \pi t, & 0 \leq t \leq 1 & \quad \mathbf{3/8} \end{aligned}$$

Do these signals have the equal energy? What is the
average energy per symbol?
Give an orthogonal basis for these three signals.

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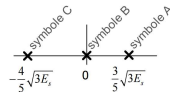
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Example: Exam1, 2013, Pr 3

Considérez le système 3PAM suivant.



Le système a $E_b/N_0 = 7.2$ dB, avec $E_s = 25/3$, et $N_0 = 1$.

- A. (10 points) Supposons que les symboles ont tous la même probabilité a priori. Quels sont les seuils de décision pour chaque symbole?
- B. (20 points) Supposons que le symbole A est 3 fois plus probable que les symboles B, et C (qui ont tous la même probabilité a priori). Quels sont les seuils de décision pour chaque symbole?

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Digital Communications

Chapitre 3 MF/correlator equivalence

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Topics covered

- Linear sampling receiver (Sec. 3.1)
- Matched filter (Sec. 3.2.2)
- Signal space et Gram-Schmidt process (Sec. 3.1 and course notes)
- Error probability & receivers MAP & ML (Sec. 3.2, appendix B, course notes) equivalences
- Modulation OOK et antipodal (Sec. 3.2)
- Intersymbol interference (Sec. 3.3, 3.4.1, 3.4.2)
- Pages 104-151

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Two approaches

- Linear sampling receiver
 - ❑ fix the type of receiver
 - ❑ probability theory for decision rule
 - ❑ maximize the SNR with a matched filter
- Signal space analysis
 - ❑ representation in a vector space
 - ❑ probability theory for decision rule
 - ❑ structure is a correlator

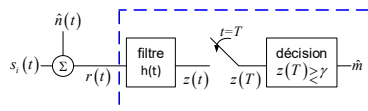
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Sampling receiver



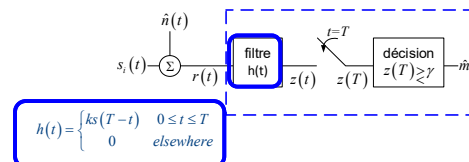
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Sampling receiver



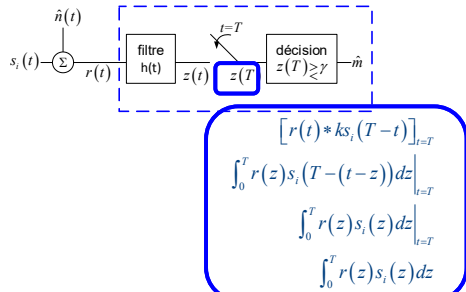
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Sampling receiver



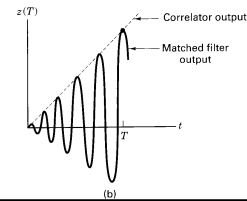
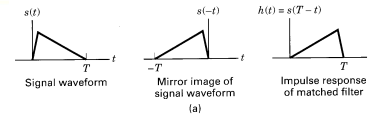
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Equivalence at $t=T$

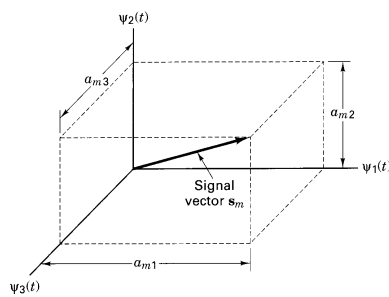


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Signal space



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Signal space receiver

➤ MAP

□ i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

➤ ML

□ i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$

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Signal space receiver

➤ MAP

□ i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

➤ ML

□ i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$

What is the structure of the test statistics??

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Signal space receiver

➤ MAP

□ i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$$

➤ ML

□ i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$$

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Signal space

doesn't change the minimisation – identical for all i

➤ MAP
 i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$
 $\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

➤ ML
 i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$
 $\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$

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Signal space receiver

known; calculated in advance – not at each reception

➤ MAP
 i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$
 $\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

➤ ML
 i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$
 $\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$

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Signal space receiver

only calculation at the receiver

➤ MAP
 i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$
 $\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

➤ ML
 i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$
 $\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$

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Signal space receiver

correlator

➤ MAP
 i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$
 $\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

➤ ML
 i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$
 $\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$

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Correlator

$\langle r(t), s_1(t) \rangle$
 $= \int_0^T r(t) s_1(t) dt$

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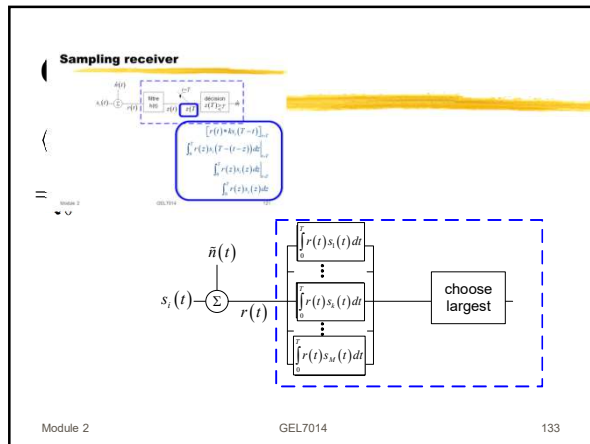
Linear sampling receiver

$h(t) = \begin{cases} ks(T-t) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$

$\left[r(t) * ks_i(T-t) \right]_{t=T}$
 $\int_0^T r(z) s_i(T-(z-t)) dz \Big|_{t=T}$
 $\int_0^T r(z) s_i(z) dz$

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Digital communications

The choice of waveforms ... the binary case

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Topics covered

- Linear sampling receiver (Sec. 3.1)
- Matched filter (Sec. 3.2.2)
- Signal space et Gram-Schmidt process (Sec. 3.1 and course notes)
- Error probability & receivers MAP & ML (Sec. 3.2, appendix B, course notes)
- Modulation OOK et antipodal (Sec. 3.2)
- Intersymbol interference (Sec. 3.3, 3.4.1, 3.4.2)
- Pages 104-151

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Noise... key for finding BER

- zero mean Gaussian

$$n_j \sim \left(0, \frac{N_0}{2}\right) \quad \sigma^2 = \frac{N_0}{2}$$
- Independent coefficients

$$E n_i n_j = 0 \quad i \neq j$$
- Probability density

$$p_{\underline{n}}(\underline{n}) = \prod_{j=1}^{\infty} p_{n_j}(n_j) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-n_j^2/2\sigma^2}$$

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Binary case

➤ Case 1 Antipodal

$$s_1(t) = \begin{cases} A & 0 \leq t \leq T \\ 0 & \text{ailleurs} \end{cases}$$

$$s_0(t) = -s_1(t)$$

➤ Cas 2 On-Off Keying

$$s_1(t) = \begin{cases} A & 0 \leq t \leq T \\ 0 & \text{ailleurs} \end{cases}$$

$$s_0(t) = 0$$

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Gram-Schmidt Process

➤ Case 1 Antipodal

$$\psi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$$

$$E_1 = \int_0^T s_1^2(t) dt = \int_0^T A^2 dt = A^2 T$$

$$\psi_1(t) = \frac{1}{A\sqrt{T}} s_1(t) = \begin{cases} 1/\sqrt{T} & 0 \leq t \leq T \\ 0 & \text{ailleurs} \end{cases}$$

➤ Cas 2 On-Off Keying

$$\psi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$$

$$E_1 = \int_0^T s_1^2(t) dt = \int_0^T A^2 dt = A^2 T$$

$$\psi_1(t) = \begin{cases} 1/\sqrt{T} & 0 \leq t \leq T \\ 0 & \text{ailleurs} \end{cases}$$

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Signal coefficients

Case 1 Antipodal

$$\begin{aligned}s_1(t) &= \sqrt{E_b} \psi_1(t) = A\sqrt{T} \psi_1(t) \\ s_0(t) &= -s_1(t) = -A\sqrt{T} \psi_1(t) \\ \underline{s}_1 &= [A\sqrt{T}] \quad \underline{s}_0 = [-A\sqrt{T}]\end{aligned}$$

Case 2 On-Off Keying

$$\begin{aligned}s_1(t) &= \sqrt{E_b} \psi_1(t) = A\sqrt{T} \psi_1(t) \\ s_0(t) &= 0 \\ \underline{s}_1 &= [A\sqrt{T}] \quad \underline{s}_0 = [0]\end{aligned}$$

Signal space is of unit dimension

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One dimensional space

Received signal $r = [r]$

$$\begin{aligned}r &= \langle r(t), \psi_1(t) \rangle = \frac{1}{A\sqrt{T}} \langle r(t), s_1(t) \rangle \\ &= \frac{1}{A\sqrt{T}} \int_0^T r(t) s_1(t) dt\end{aligned}$$

Correlator version

- Sampling receiver equivalent to matched filter

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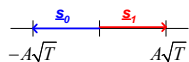
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Signal space

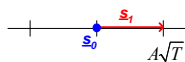
Case 1 Antipodal

$$\underline{s}_1 = [A\sqrt{T}] \quad \underline{s}_0 = [-A\sqrt{T}]$$



Cas 2 On-Off Keying

$$\underline{s}_1 = [A\sqrt{T}] \quad \underline{s}_0 = [0]$$



A fair comparison takes a normalization by the average energy of a bit

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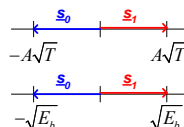
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Normalization of E_b

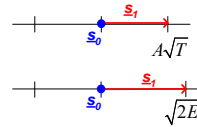
Case 1 Antipodal

$$\begin{aligned}E_1 &= \int_0^T s_1^2(t) dt = \int_0^T A^2 dt = A^2 T \\ E_0 &= \int_0^T s_0^2(t) dt = \int_0^T A^2 dt = A^2 T \\ E_b &= \frac{1}{2} E_1 + \frac{1}{2} E_0 = A^2 T\end{aligned}$$



Cas 2 On-Off Keying

$$\begin{aligned}E_1 &= \int_0^T s_1^2(t) dt = \int_0^T A^2 dt = A^2 T \\ E_0 &= \int_0^T s_0^2(t) dt = 0 \\ E_b &= \frac{1}{2} E_1 + \frac{1}{2} E_0 = A^2 T / 2\end{aligned}$$



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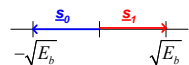
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Distance as a function on E_b

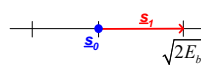
Case 1 Antipodal

- Separation is $2\sqrt{E_b}$
- Larger separation, so symbols easier to distinguish



Cas 2 On-Off Keying

- Separation is $\sqrt{2E_b}$



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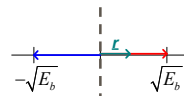
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MAP Receiver

Case 1 Antipodal

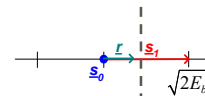
- Choose the signal closest to received signal



$$\begin{aligned}r \geq 0 &\Rightarrow \text{choose } s_1(t) \\ r \leq 0 &\Rightarrow \text{choose } s_0(t)\end{aligned}$$

Cas 2 On-Off Keying

- Choose the signal closest to received signal



$$\begin{aligned}r \geq \frac{\sqrt{E_b}}{\sqrt{2}} &\Rightarrow \text{choose } s_1(t) \\ r \leq \frac{\sqrt{E_b}}{\sqrt{2}} &\Rightarrow \text{choose } s_0(t)\end{aligned}$$

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Calculate P_e

$r \geq 0 \Rightarrow \text{choisir } s_1(t)$
 $r \leq 0 \Rightarrow \text{choisir } s_0(t)$

- Case 1 Antipodal
- Cas 2 On-Off Keying

$$P_e = P(0 \text{ sent})P(r \geq 0|0 \text{ sent}) + P(1 \text{ sent})P(r \leq 0|1 \text{ sent})$$

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Calculate P_e equal priors

- Case 1 Antipodal
- Cas 2 On-Off Keying

$$\begin{aligned} P_e &= P(0 \text{ sent})P(r \geq 0|0 \text{ sent}) + P(1 \text{ sent})P(r \leq 0|1 \text{ sent}) \\ &= \frac{1}{2}P(r \geq 0|r = -\sqrt{E_b} + n) + \frac{1}{2}P(r \leq 0|r = \sqrt{E_b} + n) \end{aligned}$$

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Calculate P_e condition on data sent

- Case 1 Antipodal
- Cas 2 On-Off Keying

$$\begin{aligned} P_e &= P(0 \text{ sent})P(r \geq 0|0 \text{ sent}) + P(1 \text{ sent})P(r \leq 0|1 \text{ sent}) \\ &= \frac{1}{2}P(r \geq 0|r = -\sqrt{E_b} + n) + \frac{1}{2}P(r \leq 0|r = \sqrt{E_b} + n) \end{aligned}$$

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Calculate P_e

- Case 1 Antipodal
- Cas 2 On-Off Keying

$$\begin{aligned} P_e &= P(0 \text{ sent})P(r \geq 0|0 \text{ sent}) + P(1 \text{ sent})P(r \leq 0|1 \text{ sent}) \\ &= \frac{1}{2}P(r \geq 0|r = -\sqrt{E_b} + n) + \frac{1}{2}P(r \leq 0|r = \sqrt{E_b} + n) \\ &= \frac{1}{2}P(-\sqrt{E_b} + n \geq 0) + \frac{1}{2}P(\sqrt{E_b} + n \leq 0) \\ &= \frac{1}{2}P(n \geq \sqrt{E_b}) + \frac{1}{2}P(n \leq -\sqrt{E_b}) \\ &= P(n \geq \sqrt{E_b}) = \int_{\sqrt{E_b}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-n^2/2\sigma^2} dn \end{aligned}$$

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Calculate P_e

- Case 1 Antipodal
- Cas 2 On-Off Keying

$$\begin{aligned} P_e &= P(0 \text{ sent})P(r \geq 0|0 \text{ sent}) + P(1 \text{ sent})P(r \leq 0|1 \text{ sent}) \\ &= \frac{1}{2}P(r \geq 0|r = -\sqrt{E_b} + n) + \frac{1}{2}P(r \leq 0|r = \sqrt{E_b} + n) \\ &= \frac{1}{2}P(-\sqrt{E_b} + n \geq 0) + \frac{1}{2}P(\sqrt{E_b} + n \leq 0) \\ &= \frac{1}{2}P(n \geq \sqrt{E_b}) + \frac{1}{2}P(n \leq -\sqrt{E_b}) \\ &= P(n \geq \sqrt{E_b}) = \int_{\sqrt{E_b}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-n^2/2\sigma^2} dn \end{aligned}$$

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Calculate P_e

- Case 1 Antipodal
- Case 2 On-Off Keying

$$\begin{aligned} P_e &= \int_{\sqrt{E_b}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-n^2/2\sigma^2} dn \quad Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2/2} dx \\ P_e &= \int_{\sqrt{E_b}/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \\ &= Q\left(\frac{\sqrt{E_b}}{\sigma}\right) = Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \end{aligned}$$

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Calculate P_e

➤ Case 1 Antipodal

➤ Case 2 On-Off Keying

$$r \geq \frac{\sqrt{E_b}}{\sqrt{2}} \Rightarrow \text{choose } s_1(t) \quad r \leq \frac{\sqrt{E_b}}{\sqrt{2}} \Rightarrow \text{choose } s_0(t)$$

$$\begin{aligned} P_e &= \frac{1}{2}P\left(r \geq \frac{\sqrt{E_b}}{\sqrt{2}} \mid r = n\right) + \frac{1}{2}P\left(r \leq \frac{\sqrt{E_b}}{\sqrt{2}} \mid r = \sqrt{E_b} + n\right) \\ &= P\left(n \geq \frac{\sqrt{E_b}}{\sqrt{2}}\right) = \int_{\frac{\sqrt{E_b}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-u^2/2\sigma^2} du \\ &= Q\left(\frac{\sqrt{E_b}/2}{\sigma}\right) = Q\left(\frac{\sqrt{E_b}/2}{\sqrt{N_0/2}}\right) = Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0}}\right) \end{aligned}$$

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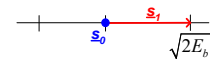
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OOK vs. BPSK

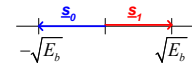
➤ OOK: $s_1(t)$ NRZ, $s_0(t) = 0$

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$



➤ BPSK: $s_1(t)$ NRZ, $s_0(t) = -s_1(t)$

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$



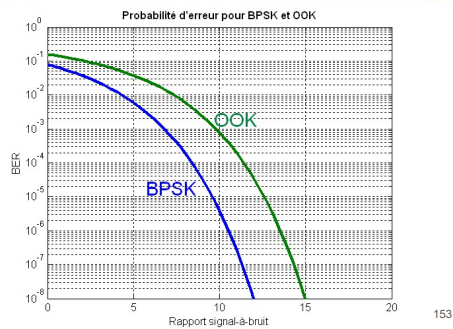
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Probability of error



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Error function (erf)

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-v^2/2} dv \leq \frac{1}{2} e^{-x^2/2} \quad \text{for } x > 3$$

Matlab $q = .5 - .5 * \text{erf}\left(\frac{x}{\sqrt{2}}\right)$

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Avoiding ISI with Nyquist pulses

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Topics covered

- Linear sampling receiver (Sec. 3.1)
- Matched filter (Sec. 3.2.2)
- Signal space et Gram-Schmidt process (Sec. 3.1 and course notes)
- Error probability & receivers MAP & ML (Sec. 3.2, appendix B, course notes)
- Modulation OOK et antipodal (Sec. 3.2)
- Intersymbol interference (Sec. 3.3, 3.4.1, 3.4.2)
- Pages 104-151

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Some observations on the matched filter ...

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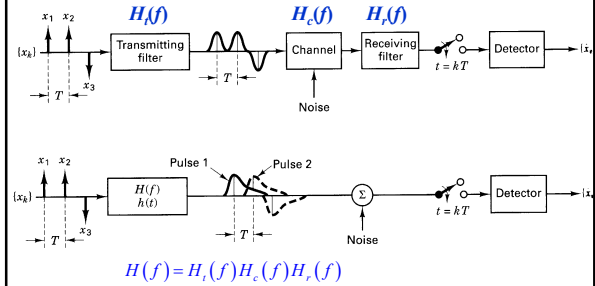
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Effective filter

$$\text{matched filter} \\ H_r(f) = H_i^*(f) H_c(f)$$



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Matched filter

$$H(f) = H_i(f) H_c(f) H_r(f)$$

- System
 - the receiver sees the TX + channel
 - The receiver must be adapted to the TX + canal
- Channel
 - maybe unknown
 - we want a general solution
 - perfect channel – an all-pass channel

$$H(f) = H_i(f) H_r(f)$$

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Quelle forme d'onde pour le signal?

- Théorie
 - toute forme d'onde peut maximiser le SNR
 - tout équivalent dans ce sens
 - en supposant que le canal n'écrase pas le signal
- Systèmes pratiques
 - composants imparfaits
 - canaux qui causent ISI

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Intersymbol interference

How to avoid it ...

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Intersymbol interference

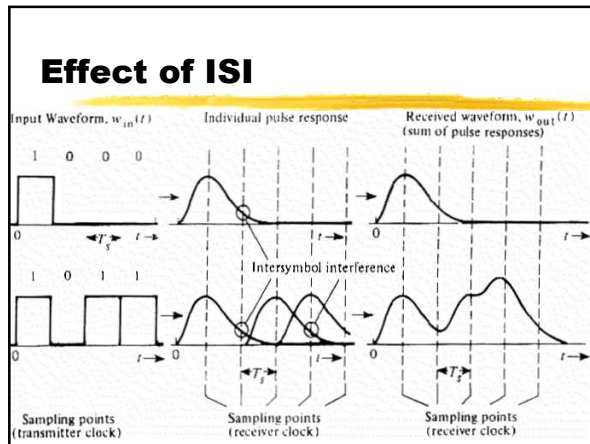
- Channel always bandlimited (physical)
- Channel always time limited (physical)
- But...
 - A time-limited signal is never bandlimited (mathematics)
 - We are always filtering a time-limited signal with a bandlimited channel
- Pulse stretching
 - Overlapping of a pulse by the previous pulse(s)

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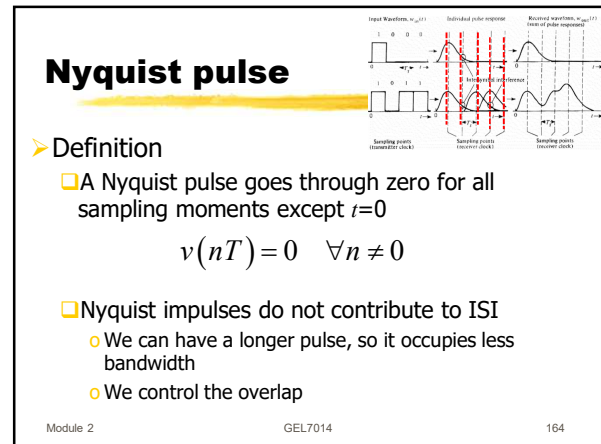
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Definition

- A Nyquist pulse goes through zero for all sampling moments except $t=0$

$$v(nT) = 0 \quad \forall n \neq 0$$

- Nyquist impulses do not contribute to ISI

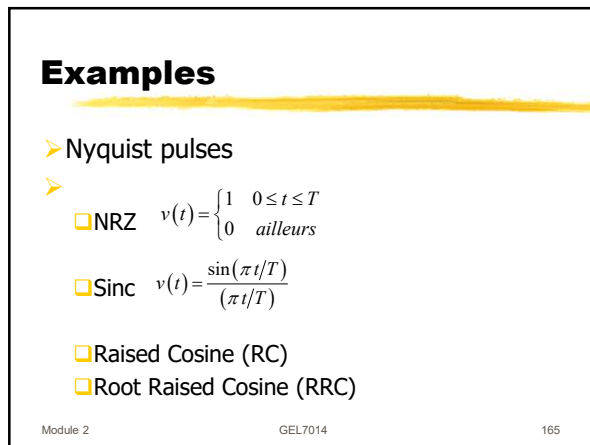
- We can have a longer pulse, so it occupies less bandwidth
- We control the overlap

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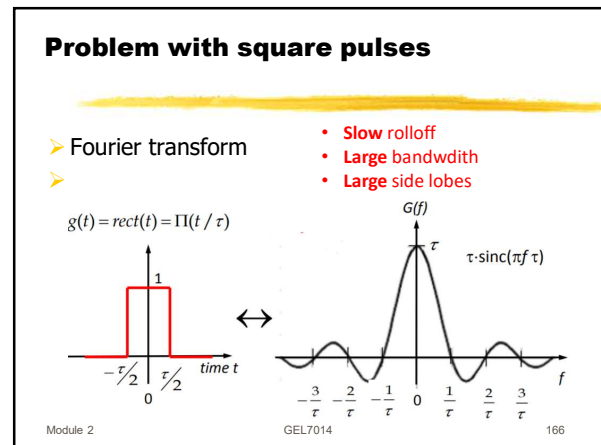


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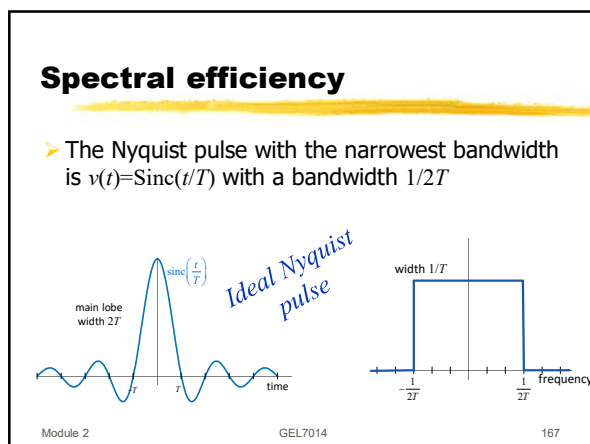


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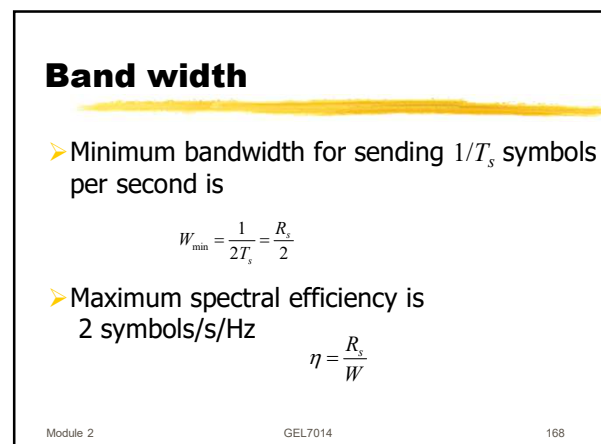


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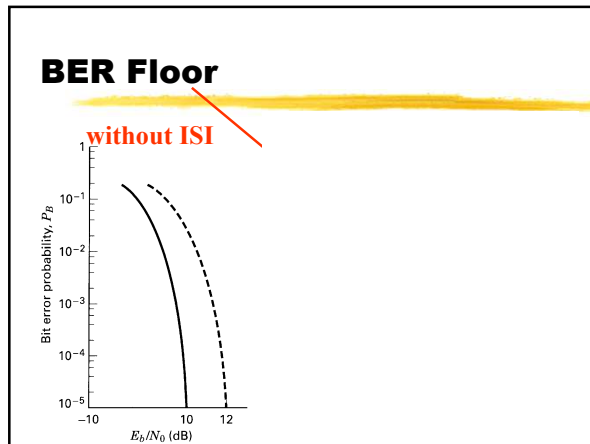


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Digital Communications

Chapitre 3

Raised Cosine pulses

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Compromise

- Desirable impulses
 - Spectrally efficient
 - Avoids ISI
 - Practical implementation
 - robust to timing errors

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Raised Cosine

- Pulse
$$v(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s} \frac{\cos(r\pi t/T_s)}{1 - 4r^2 t^2/T_s^2}$$
- Spectrum
$$V(f) = \begin{cases} 1 & 0 < |f| < \frac{1-r}{2T_s} \\ \cos^2 \left[\frac{\pi T_s}{2r} \left(f - \frac{1-r}{2T_s} \right) \right] & \frac{1-r}{2T_s} < |f| < \frac{1+r}{2T_s} \\ 0 & \text{elsewhere} \end{cases}$$

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Raised Cosine

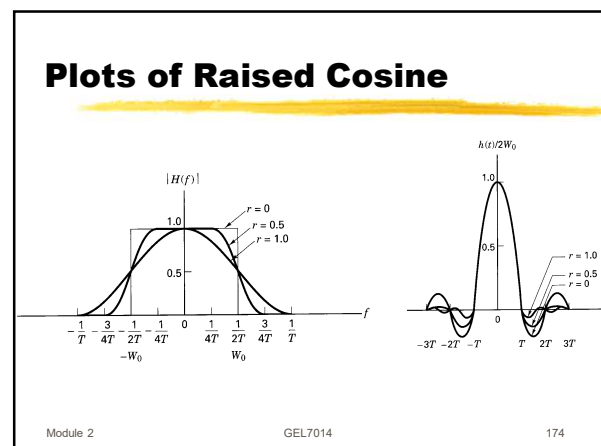
sinc function

- Impulsion
$$v(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s} \frac{\cos(r\pi t/T_s)}{1 - 4r^2 t^2/T_s^2}$$
- Spectre
$$V(f) = \begin{cases} 1 & 0 < |f| < \frac{1-r}{2T_s} \\ \cos^2 \left[\frac{\pi T_s}{2r} \left(f - \frac{1-r}{2T_s} \right) \right] & \frac{1-r}{2T_s} < |f| < \frac{1+r}{2T_s} \\ 0 & \text{elsewhere} \end{cases}$$

rolloff factor

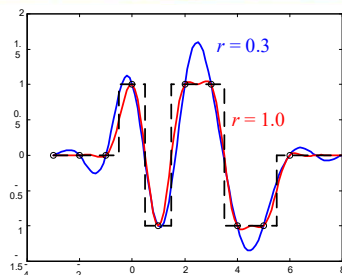
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Bit sequence



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Root Raised Cosine

➤ Define

$$V_{RRC}(f) = \sqrt{V_{RC}(f)} \quad \text{ou} \quad V_{RRC}^2(f) = V_{RC}(f)$$

➤ Use

$$H_t(f) = H_r(f) = V_{RRC}(f)$$

$$H_t(f)H_r(f) = V_{RRC}^2(f) = V_{RC}(f)$$

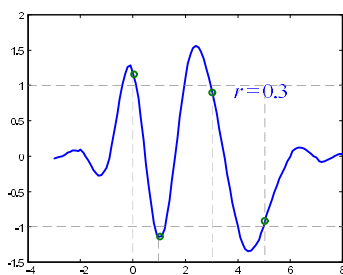
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Root Raised Cosine



not a Nyquist pulse ...

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Demo Matlab

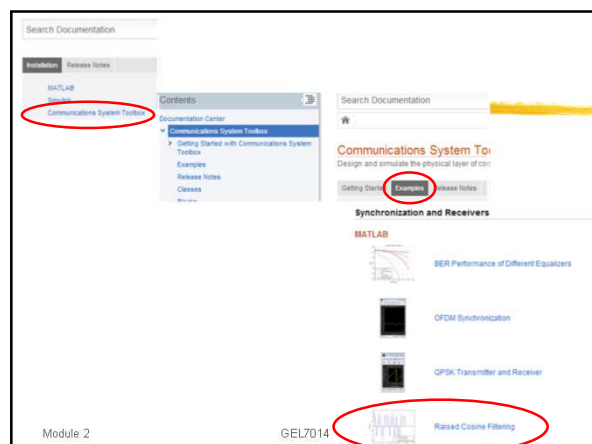
Root Raised Cosine

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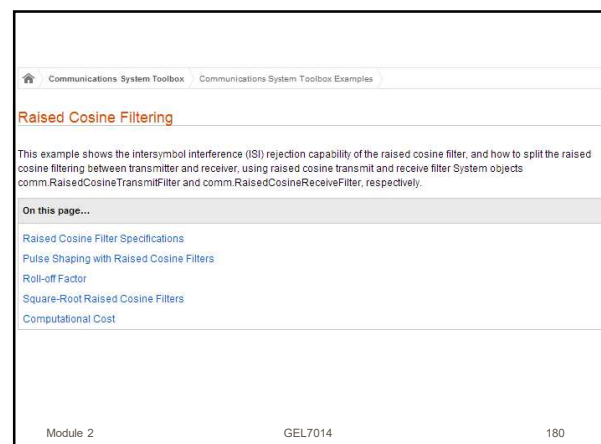
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Matlab Commands

- `randi` *generate bits randomly*
- `comm.RaisedCosineTransmitFilter` *generate a RC Filter with rolloff and specified delay*
- `step` *generate the filter output for a given input*

RRC also generated by `rcosine`

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```
Nsym = 6;           % Filter span in symbol durations
beta = 0.5;         % Roll-off factor
sampsPerSym = 8;    % Upsampling factor

%%
% We use a raised cosine transmit filter System object and set its
% properties to obtain the desired filter characteristics. We also use
% |fvtool| to visualize filter characteristics.

rcosFilt = comm.RaisedCosineTransmitFilter(...
    'Shape',         'Normal', ...
    'RolloffFactor', beta, ...
    'FilterSpanInSymbols', Nsym, ...
    'OutputSamplesPerSymbol', sampsPerSym)
```

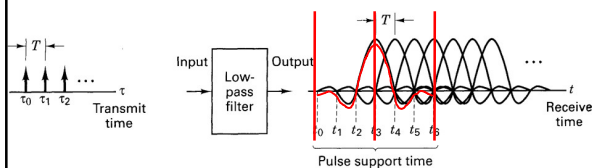
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Delay for a range of 6



- truncated with 4 sidelobes
- three bit interval delay

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```
Nsym = 6;           % Filter span in symbol durations
beta = 0.5;         % Roll-off factor
sampsPerSym = 8;    % Upsampling factor

%%
% We use a raised cosine transmit filter System object and set its
% properties to obtain the desired filter characteristics. We also use
% |fvtool| to visualize filter characteristics.

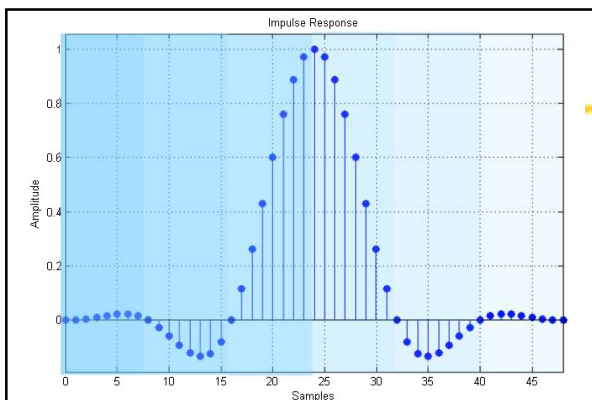
rcosFilt = comm.RaisedCosineTransmitFilter(...
    'Shape',         'Normal', ...
    'RolloffFactor', beta, ...
    'FilterSpanInSymbols', Nsym, ...
    'OutputSamplesPerSymbol', sampsPerSym)
```

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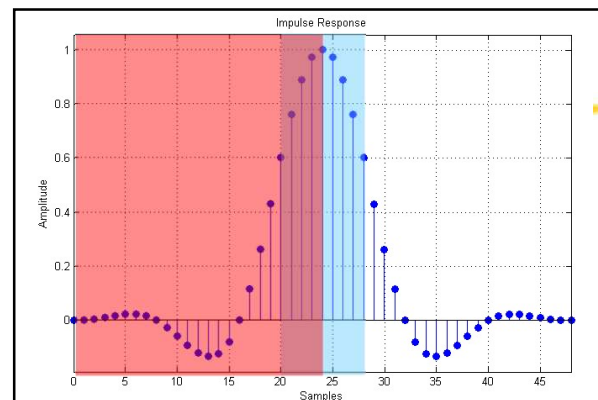


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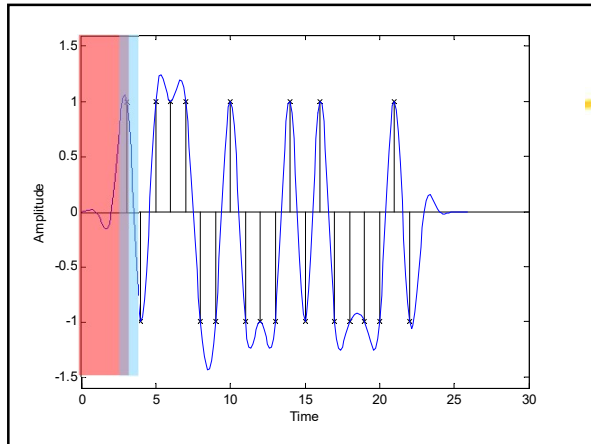


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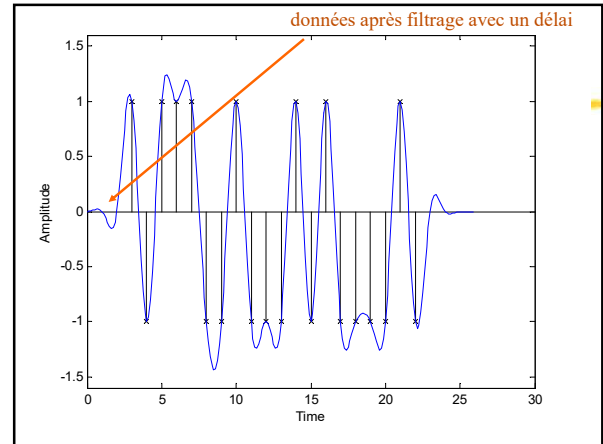
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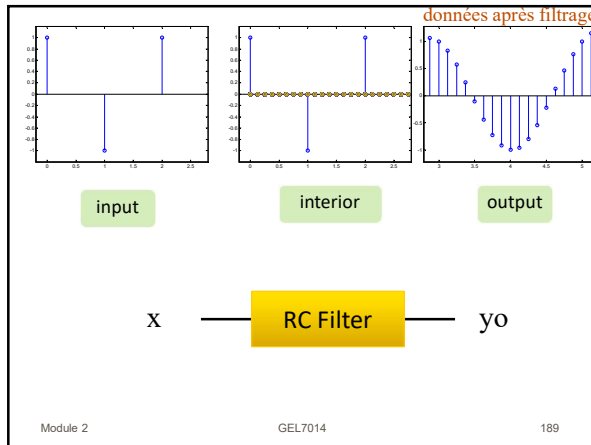
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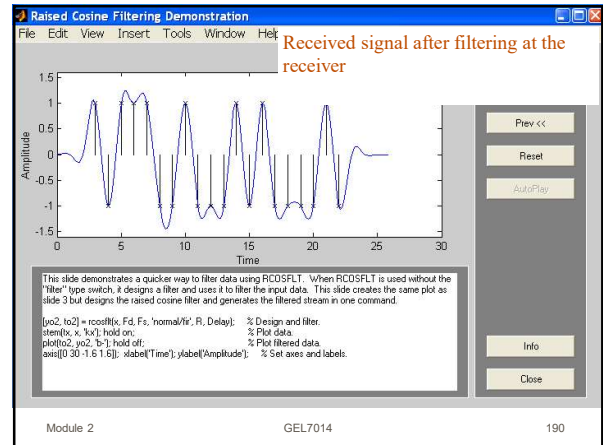
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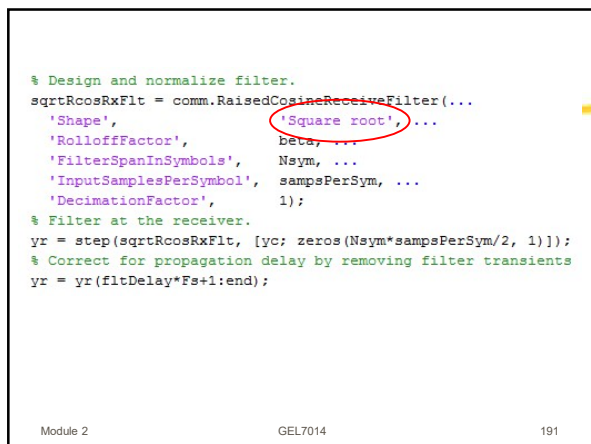
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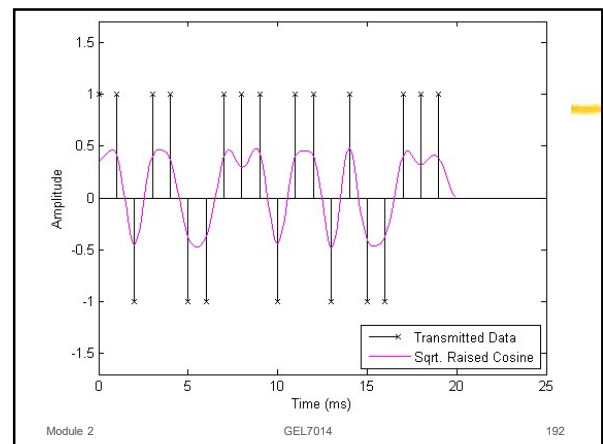
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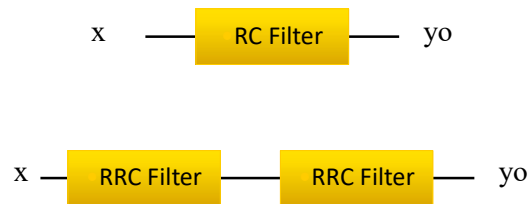


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Matlab, 2



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Noise

- Add noise in Matlab

```
signal_plus_bruit
= AWGN(signal_sans_bruit,SNR_en_dB,'measured')
```



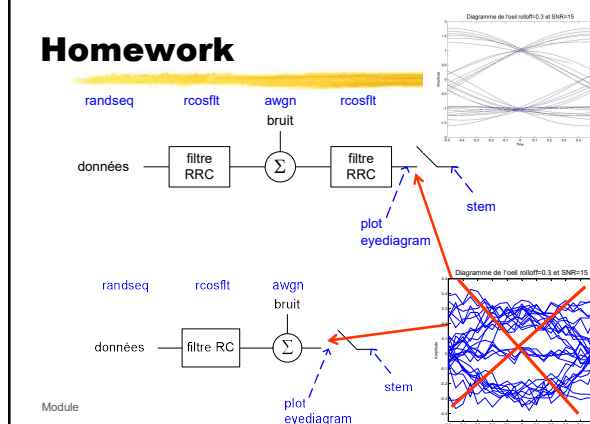
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Homework



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