

GEL7114

Module 3

Modulation

1

Modulation

- Motivation
- Many modulation formats
 - PSK, FSK, ASK, QAM
- Important criteria
 - Spectral efficiency
 - BER vs. SNR
 - Complexity (Cost)

2

Demodulation / Detection

- Two types
 - Coherent detection
 - phase of the carrier known to the receiver
 - optimal receptor: correlator
 - Gaussian noise
 - Noncoherent detection
 - carrier phase unknown at receiver
 - optimal receiver : power detection (envelope detection)
 - non-Gaussian noise

3

Topics covered

- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4)
- Probability of error & the union bound (course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M -FSK (Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Minimum spacing for M -FSK (Sec. 4.5.4)
- Gray code (Sec. 4.9.4)

4

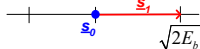
Signal space

- Binary case
 - Signal space - with good normalization
 - Distance between the two bits
 - Probability of error
- M-ary Case
 - Signal space
 - Distance between symbols

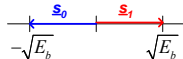
5

OOK vs. BPSK

- OOK: $s_1(t)$ NRZ, $s_0(t) = 0$

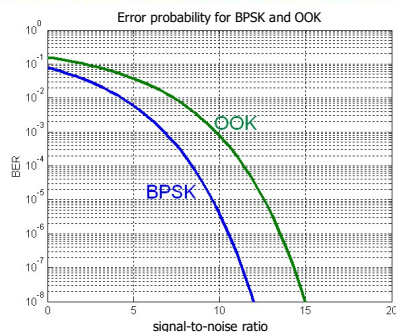
$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$


- BPSK: $s_1(t)$ NRZ, $s_0(t) = -s_1(t)$

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$


6

Error probability



7

GEL7114

Modulation formats

8

Topics covered, Part 1

- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4)
- Probability of error & the union bound (course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M -FSK (Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Gray code (Sec. 4.9.4)



9

Carrier

- Base band
 - no carrier
 - positive frequencies bandwidth
- Pass band
 - with carrier
 - bandwidth twice the base band

10

Modulation

- Reasons for carrier modulation
 - Antennas, FDMA, etc.
- Signal space analysis
 - Valid for carrier modulation
 - Basis vectors take into account modulation
 - Often $\psi_1 = \cos(\omega_0 t)$ $\psi_2 = \sin(\omega_0 t)$
 - Parties **in-phase** et **quadrature**

11

PSK

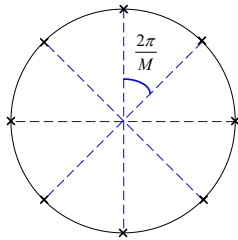
2 dimensional space no matter how many symbols you have

- Phase shift keying $s(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \phi_i(t))$ $0 \leq t \leq T$
- Typically $\phi_i = \frac{2\pi i}{M}$ $i = 1, 2, \dots, M$
- E energy/symbol, T symbol time
- Typically $\psi_1 = \cos(\omega_0 t)$ $\psi_2 = \sin(\omega_0 t)$
- Example: Antipodal (BPSK)

$\begin{array}{ccc} \times & & \times \\ -\sqrt{E_b} & & \sqrt{E_b} \end{array} \quad \psi$

12

M-PSK



- Same distance from origin for the entire constellation
- Impact of larger M ???

13

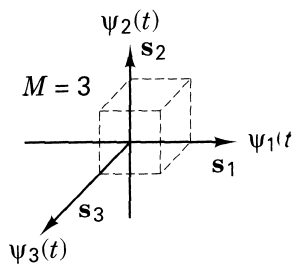
FSK

espace de dimension M

- Frequency shift keying $s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \theta) \quad 0 \leq t \leq T$
- ω_i are M distinct frequencies
- E energy/symbol, T symbol time
- Typically $\{\psi_i = \cos(\omega_i t)\}_{i=1}^M \quad \theta = 0$
- Typically M is a power of 2

14

3-FSK



- Number of frequencies M
- Dimension of space is M
- Impact of larger M ???

15

ASK = PAM

- Amplitude shift keying

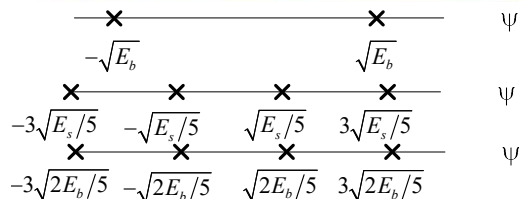
$$s_i(t) = \sqrt{\frac{2E_i}{T}} \cos \omega_c t \quad i = 1, 2, \dots, M$$

- Typically discrete levels
- Example: OOK
- Not high performance, so less popular

16

QASK

One dimensional space no matter the number of symbols



- Symbols are NOT at the same distance to the origin
- Impact of larger M ???

17

APK or QAM

2 dimensional space no matter how many symbols you have

- Amplitude-phase modulation or Quadrature amplitude modulation

$$s_i(t) = \sqrt{\frac{2E_i}{T}} \cos(\omega_c t + \phi_i(t)) \quad i = 1, 2, \dots, M$$

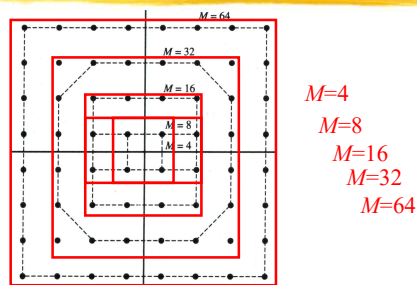
unequal energies

- Discrete levels
- Very BW efficient

$$\psi_1 = \cos(\omega_c t) \quad \psi_2 = \sin(\omega_c t)$$

18

QAM



19

Plan

- Modulation
 - PSK
 - FSK
 - ASK
 - QAM
- Scenarios
 - Binary
 - M-ary
 - Coherent detection
 - Noncoherent detection

20

Part 1

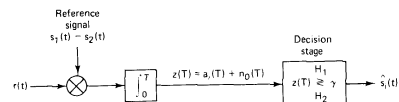
- Modulation
 - PSK
 - FSK
 - ASK
 - QAM
- Scenarios
 - Binary
 - M-ary
 - Coherent detection
 - Noncoherent detection

*Error probability already calculated
with signal space analysis*

21

Coherent detection

- Definition
- Modulations that use
 - PSK, FSK
- Form of receiver for binary case



22

Part 2

- Modulation
 - PSK
 - FSK
 - ASK
 - QAM
- Scenarios
 - Binary
 - M-ary
 - Coherent detection
 - Noncoherent detection

*Error probability will be calculated with
the union bound*

23

Part 3

- Modulation
 - PSK
 - FSK
 - ASK
 - QAM
- Scenarios
 - Binary
 - M-ary
 - Coherent detection
 - Noncoherent detection

*Error probability will be calculated with
non-Gaussian noise*

24

GEL7114

Manipulations in signal space quaternary examples

25

Signal space

- Symbols, received signal, and noise
 - Vectors in an orthonormal basis
 - Collection of basis vectors ψ_i
 - Example

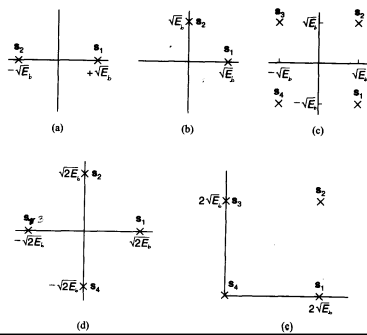
$$\underline{s} = [s_1, s_2, \dots, s_N]$$

$$s_i = \int s(t) \psi_i(t) dt$$
- Modulus squared of vector
 - Signal energy

26

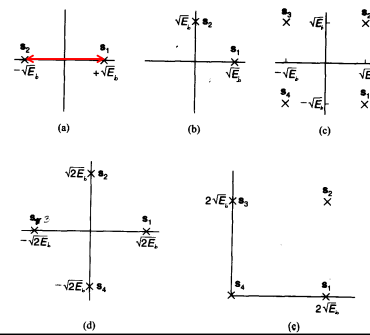
Examples

How should we look at these geometries?



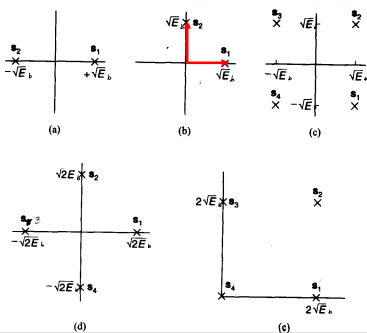
27

Examples – *antipodal*, BPSK



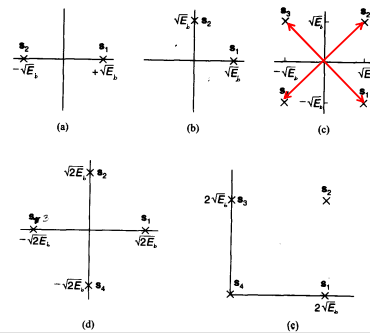
28

Examples – *orthogonal*, BFSK



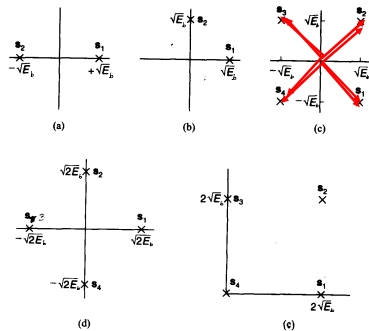
29

Examples – *antipodal*, QPSK



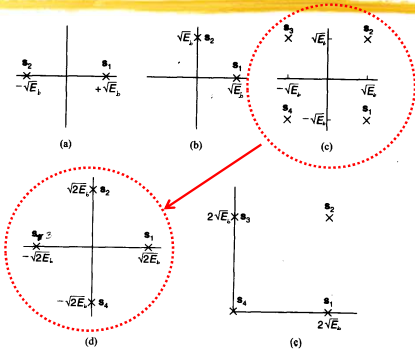
30

Examples – orthogonal, QPSK



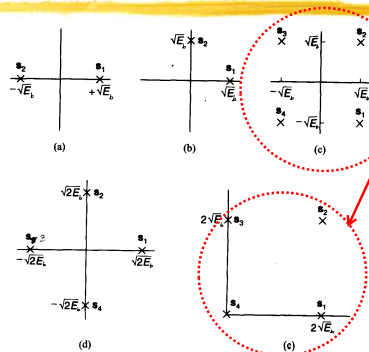
31

Examples - rotation



32

Examples - translation



33

Binary examples – Chapter 3

• Antipodal



$$D = 2\sqrt{E_b}$$

$$P_e = Q\left(\frac{D}{\sqrt{2N_0}}\right) = Q\left(\sqrt{2E_b/N_0}\right)$$

• OOK



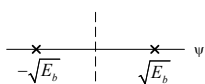
$$D = \sqrt{2E_b}$$

$$P_e = Q\left(\frac{D}{\sqrt{2N_0}}\right) = Q\left(\sqrt{E_b/N_0}\right)$$

34

Binary examples (2)

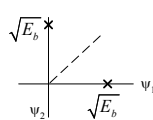
• BPSK



$$D = 2\sqrt{E_b}$$

$$P_e = Q\left(\sqrt{2E_b/N_0}\right) = Q\left(\frac{D}{\sqrt{2N_0}}\right)$$

• BFSK



$$D = \sqrt{2E_b}$$

$$P_e = Q\left(\sqrt{E_b/N_0}\right) = Q\left(\frac{D}{\sqrt{2N_0}}\right)$$

35

Minimum distance

normalization is critical !

• Definitions

$$D_{\min} = \min_{i \neq k} \|s_i - s_k\| \quad d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}}$$

➤ closer signals => more errors

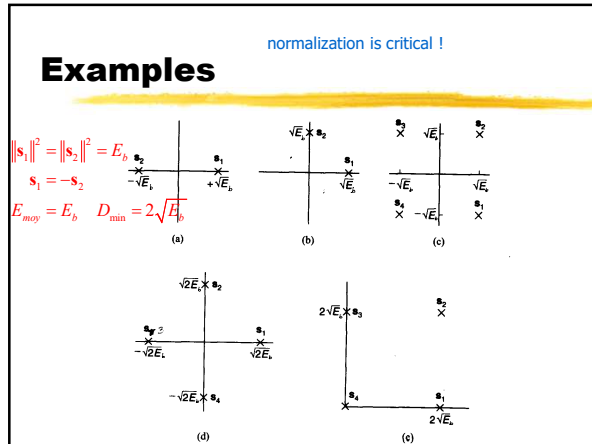
• Average energy

$$E_s = E_{\text{avg}} = \frac{1}{M} \sum_{i=1}^M \|s_i\|^2 = \frac{1}{M} \sum_{i=1}^M [\text{energy of signal } i]$$

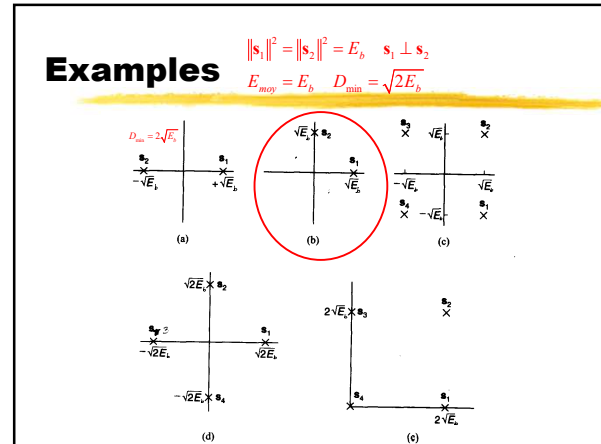
• Energy per bit vs. energy per symbol

$$E_b \log_2 M = E_s$$

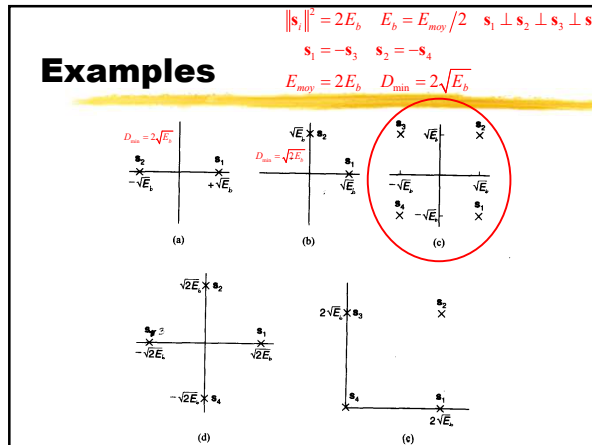
36



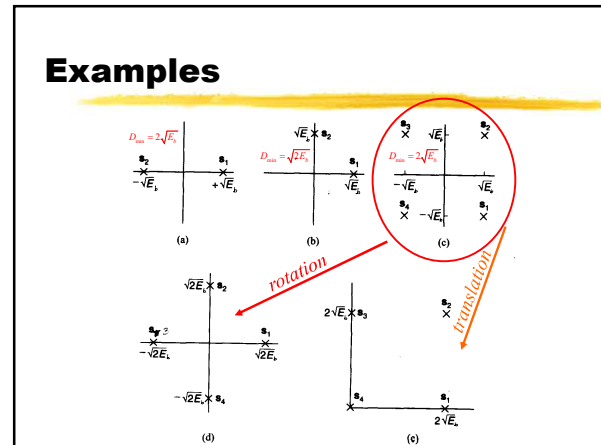
37



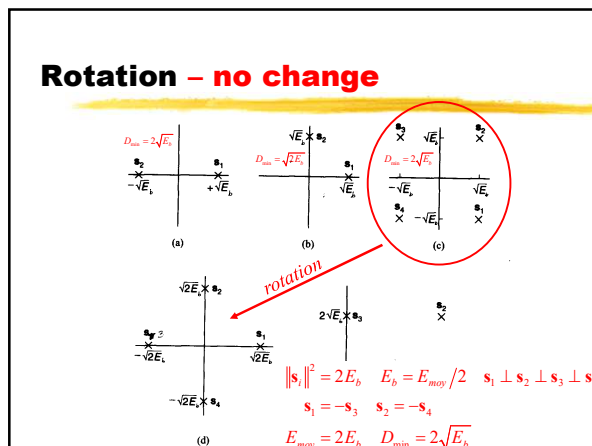
38



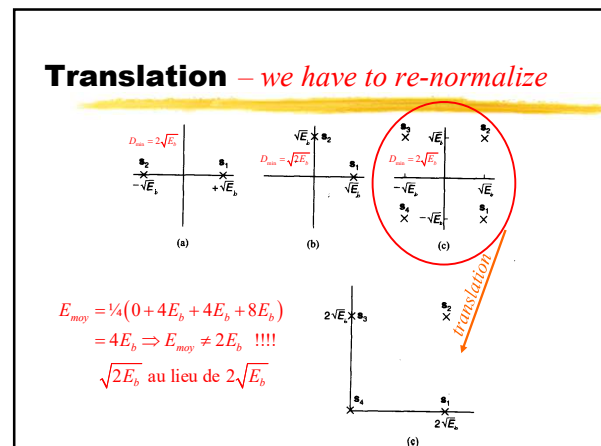
39



40

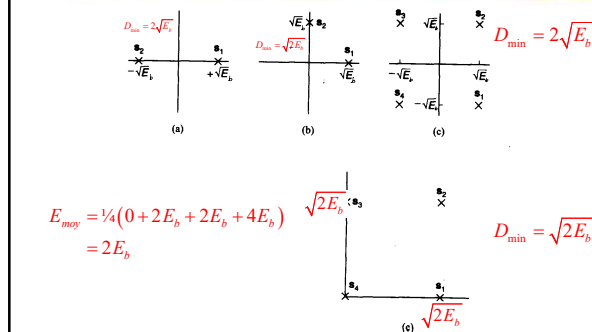


41



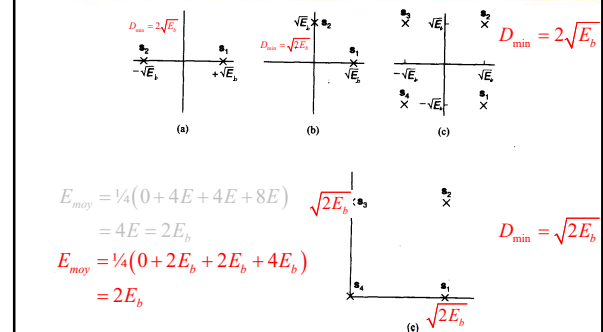
42

Translation – we have to re-normalize



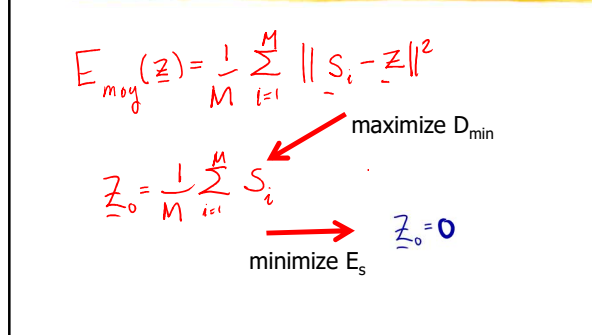
43

Translation – we have to re-normalize



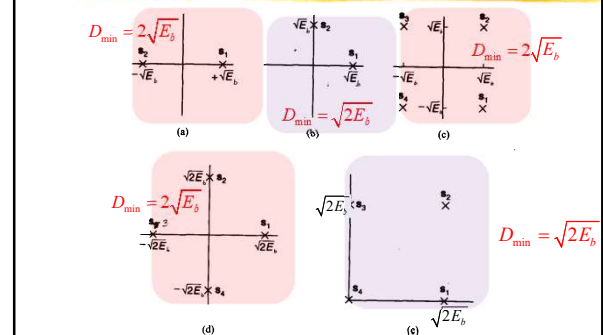
44

“Best” Translation



45

Examples



46

GEL7114

Test statistics

47

Challenges for M-ary cases

- Choosing the symbol
 - MAP most probable (*a posteriori*), and ML most likely
 - determines the form of the receiver
 - determines the decision algorithm in the receiver
- Analyzing the performance
 - exploit the results for the binary case
 - exploit the geometric interpretation of the receiver

48

Receivers

- Maximizes *a posteriori* probability (with *a priori* information)
 - i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$
- Maximizes the likelihood ratio
 - i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$

49

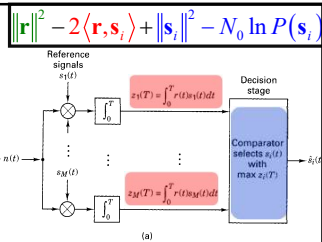
Receivers

- MAP
 - i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$
- ML
 - i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$

50

Receivers

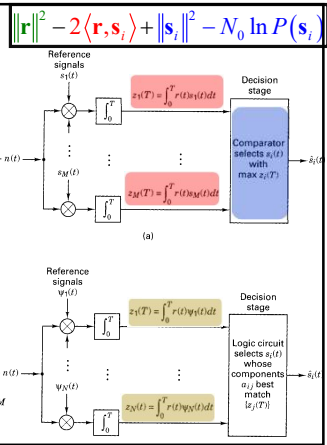
- One branch for each of M symbols



51

Receivers

- One branch for each of M symbols
- One branch for each of N basis vectors $N \leq M$



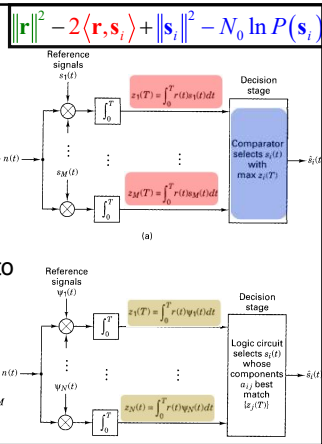
52

Receivers

- M correlations via integrals

$$\langle \mathbf{r}, \mathbf{s}_i \rangle = \int_0^T r(t)s_i(t)dt$$
- $N (\leq M)$ correlations to find $\{\mathbf{r}_j\}_{j=1}^N$

$$\langle \mathbf{r}, \mathbf{s}_i \rangle = \sum_{j=1}^N r_j s_{i,j}$$

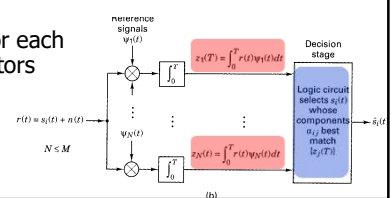


53

Receivers

- One branch for each of N basis vectors $N \leq M$

$$\|\mathbf{r} - \mathbf{s}_i\|^2 = \sum_{j=1}^N (r_j - s_{i,j})^2$$



54

Receiver vs. modulation format

- MPSK
 - two branches: in-phase et quadrature
- MFSK
 - M branches: one for each symbol
 - basis vector = symbol
 - **orthogonal modulation**
- QAM
 - two branches: in-phase and quadrature

55

Receivers

- One branch for each M symbols

orthogonal modulation

MFSK

orthogonal modulation

- Two branches

QAM

MPSK

non-orthogonal modulation

GEL7114

Decision regions

57

Topics covered, Part 1

- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4)
- Probability of error & the union bound (course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M -FSK (Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Gray code (Sec. 4.9.4)



58

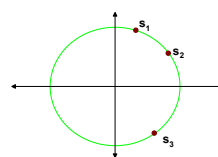
Decision algorithms

- Geometric interpretation
 - To reduce the complexity of the decision algorithm
 - To provide an intuition of how the receiver works
- i that minimizes $\|\mathbf{r} - \mathbf{s}_i\|^2$
 - Equivalent to choosing the nearest symbol
 - Regions can be determined in advance to simplify the receiver

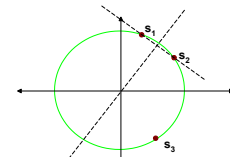
59

How to choose the nearest symbol directly from vector received?

Decision regions



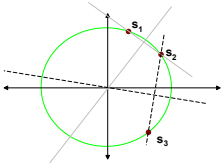
Constellation (equal energies)



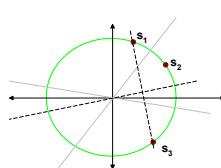
Perpendicular bisector between $\mathbf{s}_1$ and $\mathbf{s}_2$

60

Decision regions



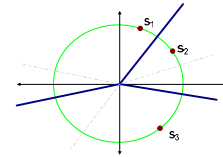
perpendicular bisector
between s_3 and s_2



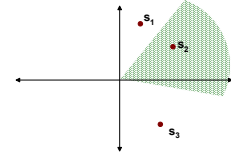
perpendicular bisector
between s_1 and s_3

61

How to choose the nearest symbol directly from vector received? $\Rightarrow$ regions



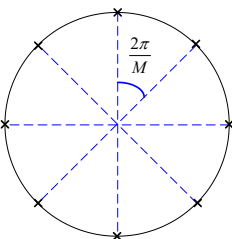
bissecteurs perpendiculaires



régions

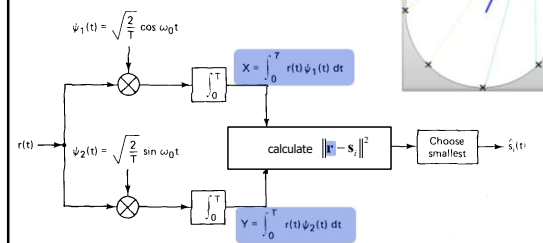
62

M-PSK



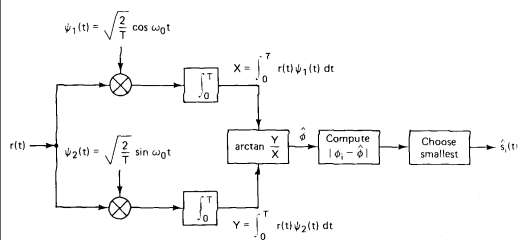
63

MPSK receiver



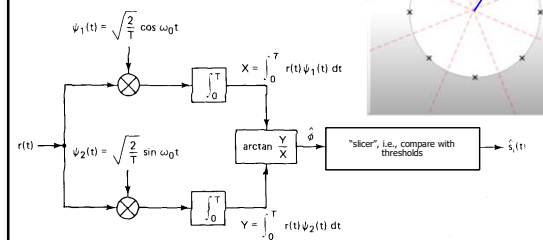
64

MPSK receiver



65

MPSK receiver



66

Challenges for M-ary cases

- Choosing the symbol
 - MAP most probable (*a posteriori*), and ML most likely
 - determines the form of the receiver
 - determines the decision algorithm in the receiver
- Analyzing the performance
 - exploit the results for the binary case
 - exploit the geometric interpretation of the receiver

67

binary case

P_e from the signal space

- What can we learn from the binary case?
 - Signal space: two points



68

binary case

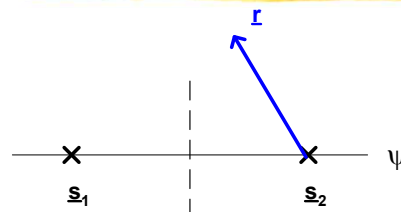
Decision regions



69

binary case

Received vector

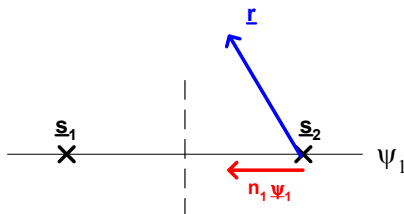


70

binary case

Noise contribution

- Region two $\Rightarrow$ choose s_2 , no error

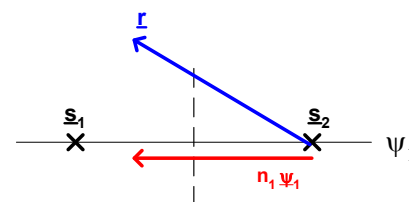


71

binary case

Another possibility

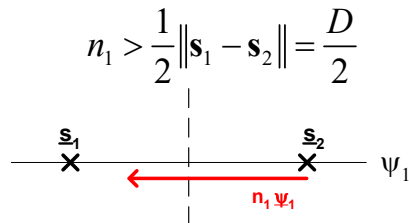
- Region one $\Rightarrow$ choose s_1 , **error**



72

Errors

- When does an error occur?
- When the noise vector exceeds the bisector ...



73

How to calculate P_e for the M-ary case now that we have the decision regions?

74

Minimal distance

- Definition $D_{\min} = \min_{i \neq k} \|s_i - s_k\|$ $d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}}$
- closer signals => more errors

75

GEL7114

Union bound

76

Topics covered, Part 1

- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4)
- The union bound & the probability of error (course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M-FSK (Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Gray code (Sec. 4.9.4)



77

QPSK

- Quadrature phase shift keying

$$s(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \phi_i(t)) \quad 0 \leq t \leq T \quad \phi_i \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$$

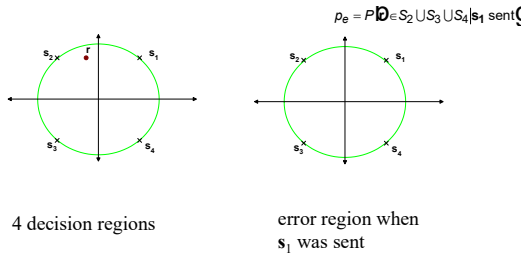
- E energy/symbol, T symbol time
- Basis vectors

$$\psi_1 = \cos(\omega_0 t) \quad \psi_2 = \sin(\omega_0 t)$$

- Decision regions – four quadrants

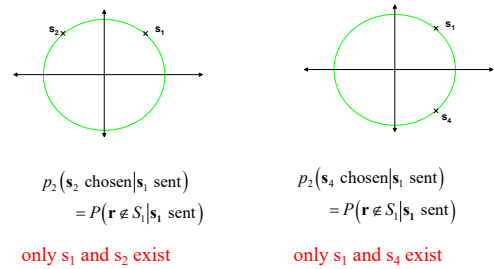
78

Union bound



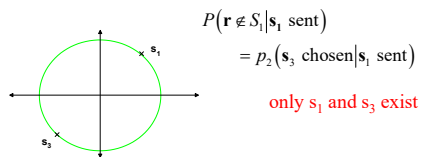
79

Union bound



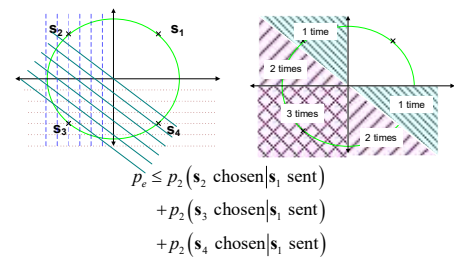
80

Union bound



81

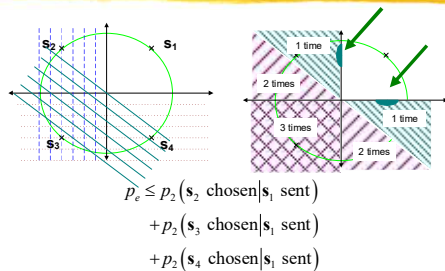
Union bound



$$P_e(QPSK) \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)$$

82

Union bound

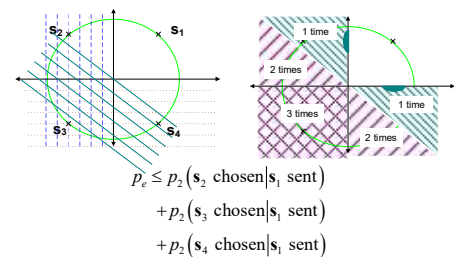


$$P_e(QPSK) \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

83

Approximation from the Union bound



$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

84

Union bound

- Error probability dominated by the nearest points

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) = \frac{2K}{M} Q\left(d_{\min} \sqrt{\frac{E_b}{N_0}}\right)$$

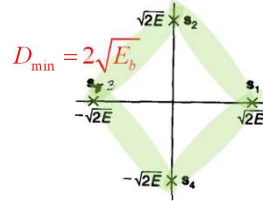
- K is the number of pairs of symbols separated by the minimum distance $D_{\min}$

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}}$$

85

QPSK

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) = \frac{2K}{M} Q\left(\frac{2\sqrt{E_b}}{\sqrt{2N_0}}\right)$$



- $M=4$
- $K=4$

$$P_e \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

86

QPSK

- Error probability coming from the union bound

$$P_e(QPSK) \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)$$

- Approximation

$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Exact error probability

$$P_e(QPSK) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

87

QPSK

0 dB

0.15963807454081

$$P_e(QPSK) \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)$$

0.15729920705029

$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

0.15111344691562

$$P_e(QPSK) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

88

QPSK

5dB

10dB

$$P_e(QPSK) \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)$$

0.01190797987629

0.00000774421643

0.01190773429556

0.00000774421643

$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

0.01187228576154

0.00000774420144

$$P_e(QPSK) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

89

Union bound

- Error probability coming from the union bound

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) = \frac{2K}{M} Q\left(d_{\min} \sqrt{\frac{E_b}{N_0}}\right)$$

- K is the number of pairs of symbols separated by the minimum distance $D_{\min}$

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}}$$

90

GEL7114

M-PSK Case

91

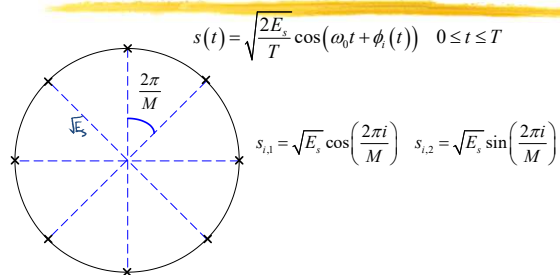
Topics covered, Part 1



- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4)
- Probability of error & the union bound
(course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M-FSK
(Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Gray

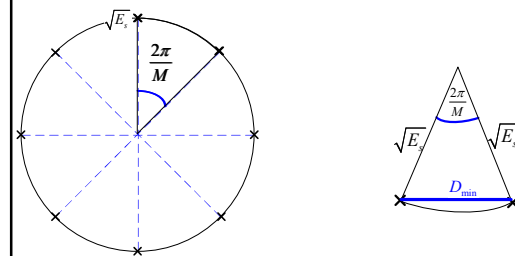
92

M-PSK



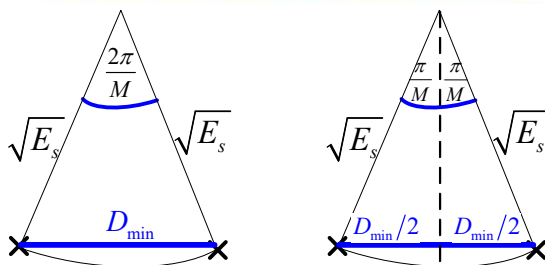
93

M-PSK



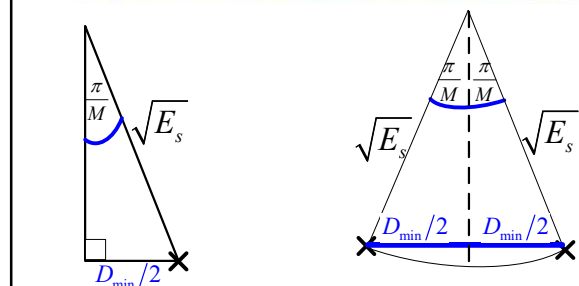
94

M-PSK



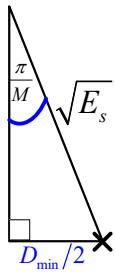
95

M-PSK



96

M-PSK



$$\sin \frac{\pi}{M} = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{D_{\min}/2}{\sqrt{E_s}} = \frac{D_{\min}}{2\sqrt{E_s}}$$

$$D_{\min} = 2 \sin \frac{\pi}{M} \sqrt{E_s} = 2 \sin \frac{\pi}{M} \sqrt{E_b \log_2 M}$$

97

M-PSK

$$P_e \approx \frac{2K}{M} Q \left(\frac{D_{\min}}{\sqrt{2N_0}} \right)$$

$$D_{\min} = 2\sqrt{E_s} \sin \frac{\pi}{M} = 2\sqrt{E_b \log_2 M} \sin \frac{\pi}{M}$$

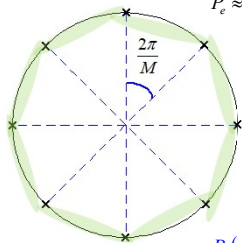
98

M-PSK

$$D_{\min} = 2\sqrt{E_b \log_2 M} \sin \frac{\pi}{M}$$

$$P_e \approx \frac{2K}{M} Q \left(\frac{D_{\min}}{\sqrt{2N_0}} \right)$$

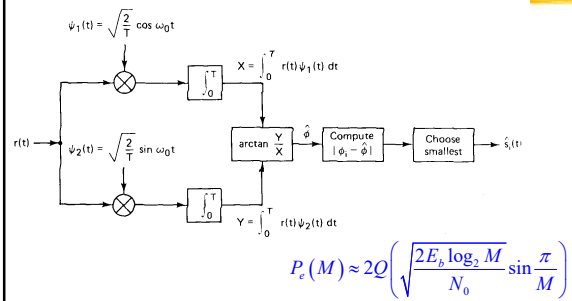
$$K = M$$



$$P_e(MPSK) \approx 2Q \left(\sin \frac{\pi}{M} \sqrt{\frac{2E_b \log_2 M}{N_0}} \right)$$

99

MPSK receiver & P_e



$$P_e(M) \approx 2Q \left(\sqrt{\frac{2E_b \log_2 M}{N_0}} \sin \frac{\pi}{M} \right)$$

100

GEL7114

M-FSK Case

101

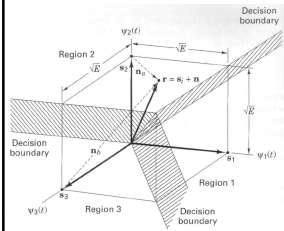
Topics covered, Part 1



- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4)
- Probability of error & the union bound (course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M-FSK (Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Gray code

102

M-FSK



$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \theta) \quad 0 \leq t \leq T$$

$$\psi_i = \sqrt{\frac{2}{T}} \cos(\omega_i t) \quad \theta = 0$$

$$\underline{s}_i = [0 \cdots \sqrt{E_s} \cdots 0]$$

i^{th} element

103

M-FSK

$$\underline{s}_i = [0 \cdots \sqrt{E_s} \cdots 0]$$

$$\underline{s}_k = [0 \cdots \sqrt{E_s} \cdots 0 \cdots 0]$$

$$\|\underline{s}_i - \underline{s}_k\| = \sqrt{\sum_{j=1}^M (s_{i,j} - s_{k,j})^2} = \sqrt{s_{i,d}^2 + s_{k,k}^2} = \sqrt{E_s + E_s} = \sqrt{2E_s}$$

- All symbol pairs are at the minimal distance

$$K = \binom{M}{2} = \frac{M!}{(M-2)!2!} = \frac{M(M-1)(M-2)!}{(M-2)!2} = \frac{M(M-1)}{2}$$

104

M-FSK

$$D_{\min} = \|\underline{s}_i - \underline{s}_k\| = \sqrt{2E_s} = \sqrt{2E_b \log M}$$

$$K = \frac{M(M-1)}{2}$$

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right)$$

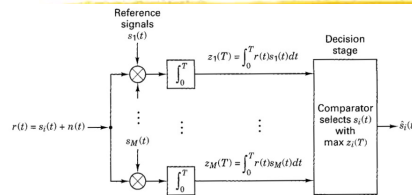
$$P_e(MFSK) \approx \frac{2M(M-1)}{2M} Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$= (M-1) Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$= (M-1) Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$

105

MFSK receiver & P_e

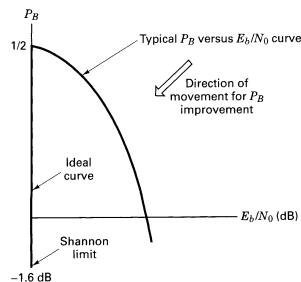


$$P_e(MFSK) \approx (M-1) Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$

106

BER curve

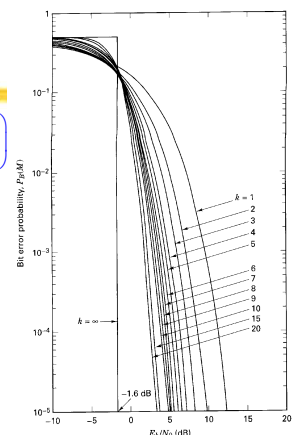
- Typical behavior
- Error probability versus SNR



107

Orthogonal M-ary

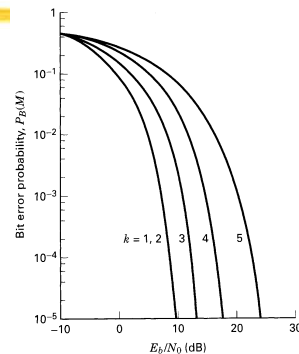
$$P_e(M) = (M-1) Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$



108

Non-orthogonal M-PSK

$$P_e(M) \approx 2Q\left(\sqrt{\frac{2E_b \log_2 M}{N_0} \sin \frac{\pi}{M}}\right)$$



109

GEL7114

QAM

Chapter 9, section 9.8.3

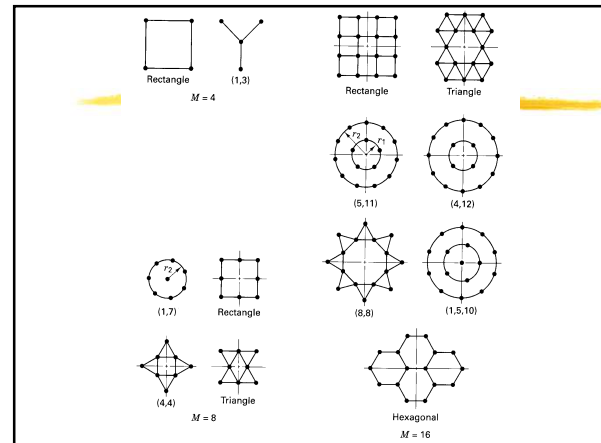
110

Topics covered, Part 1

- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4, 9.83)
- Probability of error & the union bound (course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M-FSK (Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Gray code



111



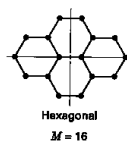
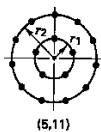
112

Average energy

- Average energy

$$E_s = E_{\text{moy}} = \frac{1}{M} \sum_{i=1}^M \|s_i\|^2 = \frac{1}{M} \sum_{i=1}^M [\text{énergie du signal } i]$$

- Energy varies from symbol to symbol for QAM



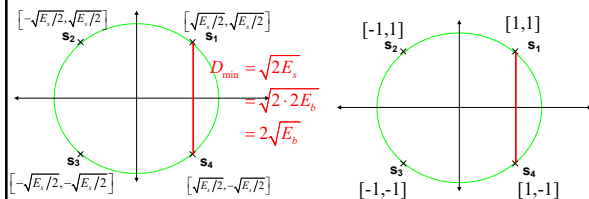
113

Coordinate systems

- Signal space
 - Normalization to ensure the distance to the origin corresponds to the signal energy
- I/Q space
 - Simple coordinates for chosen geometry
- Difference between these spaces...
 - ... a normalization

114

Example: QPSK



115

I/Q space

- MQAM – arbitrary constellation
- *Re-normalization* of the signal space

$$(\tilde{a}_n^I, \tilde{a}_n^Q) = \sqrt{\frac{M \cdot E_s}{\sum_{n=1}^M [(a_n^I)^2 + (a_n^Q)^2]}} (a_n^I, a_n^Q)$$

coordinates, signal space

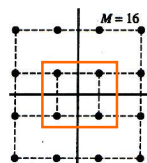
coordinates, I/Q space

116

Examples

➤ 16QAM

$(1,1)(1,-1)(-1,1)(-1,-1)$ 4 points at distance $\sqrt{2}$

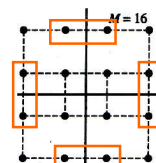


117

Examples

➤ 16QAM

$(1,1)(1,-1)(-1,1)(-1,-1)$ 4 points at distance $\sqrt{2}$
 $(3,-1)(3,1)(-3,1)(-3,-1)$ 4 points at distance $\sqrt{10}$
 $(-1,3)(1,3)(1,-3)(-1,-3)$ 4 points at distance $\sqrt{10}$

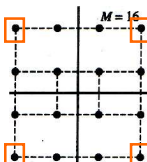


118

Examples

➤ 16QAM

$(1,1)(1,-1)(-1,1)(-1,-1)$ 4 points at distance $\sqrt{2}$
 $(3,-1)(3,1)(-3,1)(-3,-1)$ 4 points at distance $\sqrt{10}$
 $(-1,3)(1,3)(1,-3)(-1,-3)$ 4 points at distance $\sqrt{10}$
 $(3,3)(3,-3)(-3,3)(-3,-3)$ 4 points at distance $\sqrt{18}$

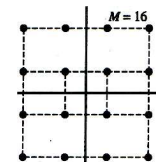


119

Examples

➤ 16QAM

$(1,1)(1,-1)(-1,1)(-1,-1)$ 4 points at distance $\sqrt{2}$
 $(3,-1)(3,1)(-3,1)(-3,-1)$ 4 points at distance $\sqrt{10}$
 $(-1,3)(1,3)(1,-3)(-1,-3)$ 4 points at distance $\sqrt{10}$
 $(3,3)(3,-3)(-3,3)(-3,-3)$ 4 points at distance $\sqrt{18}$

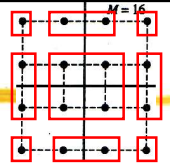


$$\sum_{n=1}^{16} (a_n^I)^2 + (a_n^Q)^2 = 4 \times 2 + 4 \times 10 + 4 \times 10 + 4 \times 18 = 160$$

120

Examples

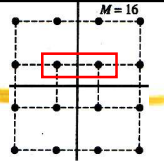
- $$(\tilde{a}_n^I, \tilde{a}_n^Q) = \sqrt{\frac{16E_s}{160}} (a_n^I, a_n^Q) = \sqrt{\frac{E_s}{10}} (a_n^I, a_n^Q)$$



121

Minimal distance

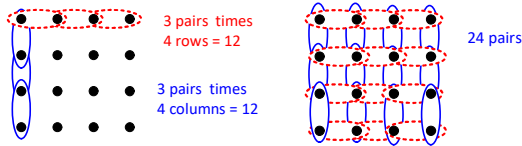
- Distance between points
 - (1,1) and (-1,1) or
 - $\sqrt{\frac{E_s}{10}}(1,1)$ et $\sqrt{\frac{E_s}{10}}(-1,1)$
- Thus
 - $D_{\min} = 2\sqrt{\frac{E_s}{10}}$
 - $E_b = \frac{E_s}{\log_2 16} = \frac{E_s}{4}$



122

Number K

- Counting pairs at the minimal distance



123

P_e of 16QAM

$$D_{\min} = 2\sqrt{\frac{E_s}{10}}$$

- Union bound

$$\begin{aligned} P_e &\approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) \\ &= \frac{2 \times 24}{16} Q\left(\sqrt{\frac{2E_s/5}{2N_0}}\right) \\ &= 3Q\left(\sqrt{\frac{E_s}{5N_0}}\right) = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) \end{aligned}$$

124

$L^2=M$ Square QAM

- For Square QAM
 - $(a_n^I, a_n^Q) \in \{\pm 1, \pm 3, \dots, \pm(L-1)\}$ L pairs
- The coordinates in signal space

$$\sqrt{\frac{3}{2(L^2-1)}} (a_n^I, a_n^Q)$$

- Minimal distance

$$\sqrt{\frac{3}{2(L^2-1)}} \sqrt{E_s} = 2\sqrt{\frac{3}{2(L^2-1)}} \sqrt{E_b \log_2 L^2} = 2\sqrt{\frac{3E_b \log_2 L}{L^2-1}}$$

125

$L^2=M$ Square QAM

$$D_{\min} = 2\sqrt{\frac{3E_b \log_2 L}{L^2-1}}$$

- Number of neighbors at the minimal distance
 - $L-1$ pairs per row times L rows
 - L pairs per column times $L-1$ columns
 - Total of $2L(L-1)$ pairs

- Estimate of the error probability

$$P_e \approx \frac{2L(L-1)}{L^2} Q\left(\sqrt{\frac{3 \log_2 M}{M-1} \frac{E_b}{N_0}}\right) = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3 \log_2 M}{M-1} \frac{E_b}{N_0}}\right)$$

126

$L^2=M$ Square QAM

- Normalized minimal distance

$$d_{\min} = \sqrt{\frac{6 \log_2 L}{L^2 - 1}} = \sqrt{\frac{3 \log_2 M}{M - 1}}$$

- Estimate of the error probability

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) = \frac{2K}{M} Q\left(d_{\min} \sqrt{\frac{E_b}{N_0}}\right) = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3 \log_2 M}{M - 1} \frac{E_b}{N_0}}\right)$$

127

$L^2=M$ Square QAM

- Approximation (Union bound)

$$P_e \approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3 \log_2 M}{M - 1} \frac{E_b}{N_0}}\right)$$

- Exact result for rectangular case (square)

$$P_e = 1 - (1 - P_m)^2$$

$$P_m = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3 \log_2 M}{M - 1} \frac{E_b}{N_0}}\right)$$

Digital Communications, Proakis

128

GEL7114

Performance vis-à-vis QPSK

129

QPSK as a reference

- Easy approach to compare modulation formats
- QPSK a reference
- Quantification of loss against a well-known reference

130

Union bound & QPSK

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) = \frac{2K}{M} Q\left(d_{\min} \sqrt{\frac{E_b}{N_0}}\right)$$

- QPSK

$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

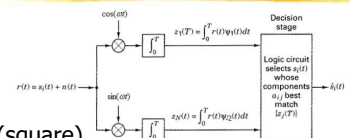
$$D_{\min} = 2\sqrt{E_b}$$

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

131

MQAM receiver & P_e

- Form of receiver



- rectangular case (square)

$$P_e \approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3 \log_2 M}{M - 1} \frac{E_b}{N_0}}\right)$$

$$P_e \approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{d_{\min}^2 E_b}{N_0}}\right) \quad d_{\min} = \sqrt{\frac{6 \log_2 L}{L^2 - 1}}$$

132

Comparison

$$P_e \approx 4 \left(1 - \frac{1}{\sqrt{M}} \right) Q \left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}} \right)$$

- Cost of enlarging the constellation

➤ QPSK $d_{\min}(L=2) = \sqrt{\frac{6}{3}} = \sqrt{2}$

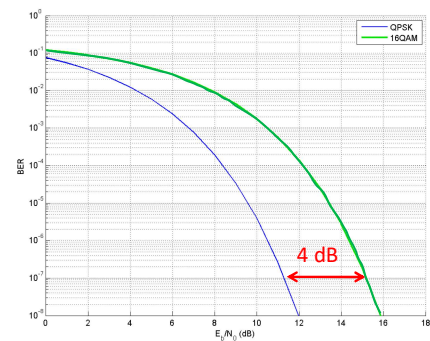
➤ 16QAM $d_{\min}(L=4) = \sqrt{\frac{6 \times 2}{15}} = \sqrt{\frac{4}{5}} = \sqrt{\frac{2}{5}} \sqrt{2}$

➤ loss $-10 \log_{10} \frac{2}{5} = 4 \text{ dB}$

$$d_{\min} = \sqrt{\frac{6 \log_2 L}{L^2 - 1}}$$

133

QPSK vs. 16QAM



134

Comparison

$$P_e \approx 4 \left(1 - \frac{1}{\sqrt{M}} \right) Q \left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}} \right)$$

- Cost of enlarging the constellation

➤ QPSK $d_{\min}(L=2) = \sqrt{\frac{6}{3}} = \sqrt{2}$

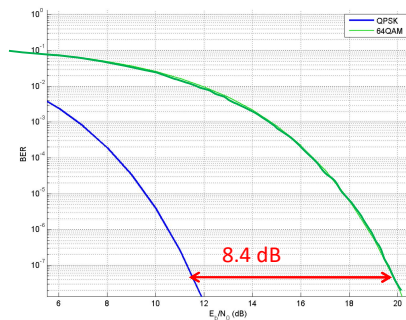
➤ 64QAM $d_{\min}(L=8) = \sqrt{\frac{6 \times 3}{63}} = \sqrt{\frac{18}{63}} = \sqrt{\frac{9}{63}} \sqrt{2}$

➤ loss $-10 \log_{10} \frac{9}{63} = 8.4 \text{ dB}$

$$d_{\min} = \sqrt{\frac{6 \log_2 L}{L^2 - 1}}$$

135

QPSK vs. 64QAM



136

Comparison

$$P_e \approx 4 \left(1 - \frac{1}{\sqrt{M}} \right) Q \left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}} \right)$$

- Cost of enlarging the constellation

➤ QPSK $d_{\min}(L=2) = \sqrt{\frac{6}{3}} = \sqrt{2}$

➤ 256QAM $d_{\min}(L=16) = \sqrt{\frac{6 \times 4}{255}} = \sqrt{\frac{24}{255}} = \sqrt{\frac{12}{255}} \sqrt{2}$

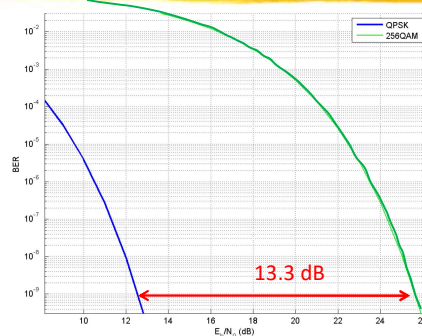
➤ loss $-10 \log_{10} \frac{12}{255} = 13.3 \text{ dB}$

loss in dB compared to QPSK

$$d_{\min} = \sqrt{\frac{6 \log_2 L}{L^2 - 1}}$$

137

QPSK vs. 256QAM



138

Example

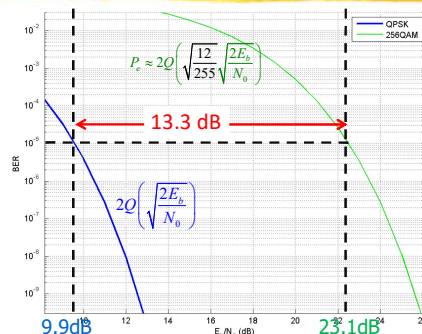
- QPSK needs $\frac{E_b}{N_0} = 9.9\text{dB}$
to reach $P_e = 10^{-5}$

- 256 QAM needs

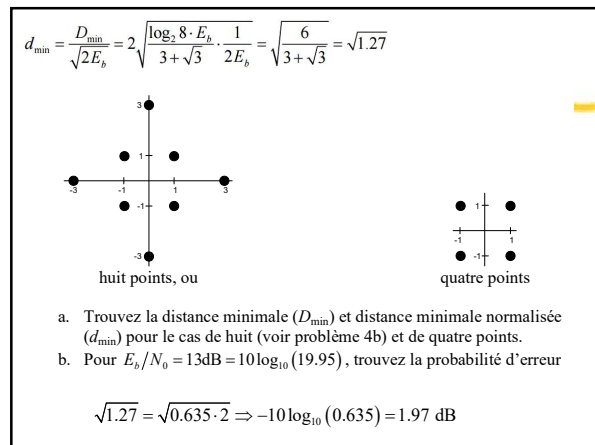
$$\frac{E_b}{N_0} = 9.9\text{dB} + 13.3\text{dB} = 23.1\text{dB}$$

139

QPSK vs. 256QAM

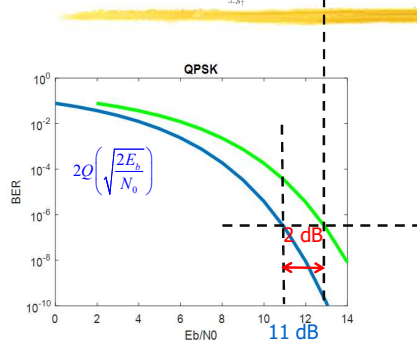


140



141

QPSK vs



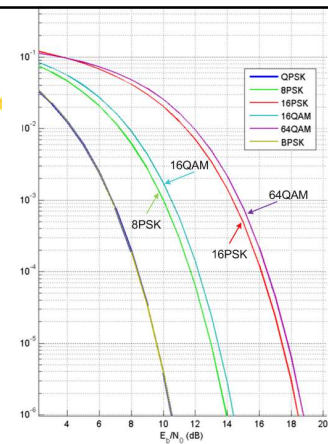
142

P_e Square QAM

- Union bound $P_e \approx 4 \left(1 - \frac{1}{\sqrt{M}}\right) Q \left(\sqrt{\frac{3 \log_2 M E_b}{M-1} \frac{E_b}{N_0}} \right)$
- Per Sklar $P_e \approx \frac{2}{\log_2 \sqrt{M}} \left(1 - \frac{1}{\sqrt{M}}\right) Q \left(\sqrt{\frac{3 \log_2 M E_b}{M-1} \frac{E_b}{N_0}} \right)$
- P_e exact $P_e = 1 - (1 - P_m)^2$ $P_m = 2 \left(1 - \frac{1}{\sqrt{M}}\right) Q \left(\sqrt{\frac{3 \log_2 M E_b}{M-1} \frac{E_b}{N_0}} \right)$

143

Probability of error



144

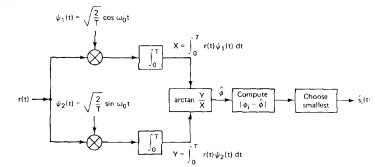
GEL7114

Examples

145

MPSK coherent receiver

- Form of receiver

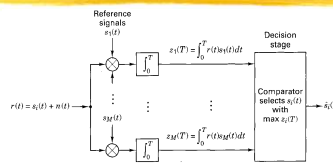


$$P_e(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{M}\right) = 2Q\left(\sqrt{\frac{2E_b \log_2 M}{N_0}} \sin \frac{\pi}{M}\right)$$

146

MPSK coherent receiver

- Form of receiver

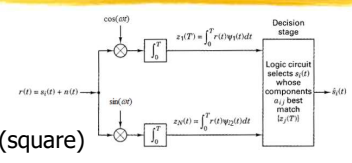


$$P_e(M) = (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right) = (M-1)Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$

147

MQAM receiver & P_e

- Form of receiver



- rectangular case (square)

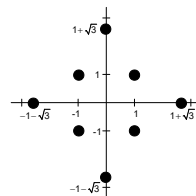
$$P_e \approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3 \log_2 M E_b}{(M-1) N_0}}\right)$$

- general case

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) = \frac{2K}{M} Q\left(d_{\min} \sqrt{\frac{E_b}{N_0}}\right)$$

148

- Considérons la constellation 8QAM donnée dans la figure suivante.



- Quelles sont les coordonnées des symboles (dans l'espace du signal)
- Trouvez la probabilité d'erreur en utilisant l'approximation venant de la borne d'union.

149

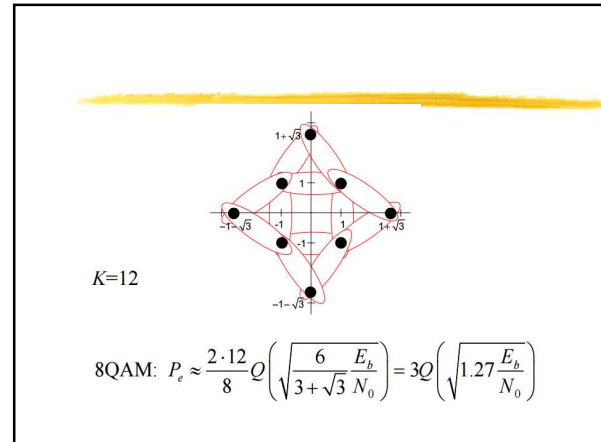
(a_n^I, a_n^Q)	# de points	distance ² de l'origine	Sous-total
$(\pm 1, \pm 1)$	4	2	8
$(0, \pm 1 \pm 3)$ $(\pm 1 \pm 3, 0)$	4	$4 + 2\sqrt{3}$	$16 + 8\sqrt{3}$
$\sum_{i=1}^M [(a_n^I)^2 + (a_n^Q)^2]$			$24 + 8\sqrt{3} = 37.86$

150

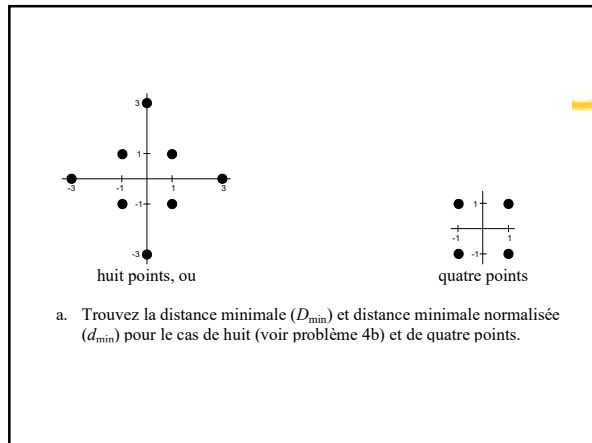
$$(\tilde{a}_n^I, \tilde{a}_n^Q) = \sqrt{\frac{8E_s}{24+8\sqrt{3}}} (a_n^I, a_n^Q) = \sqrt{\frac{E_s}{3+\sqrt{3}}} (a_n^I, a_n^Q) = \sqrt{\frac{E_s}{4.73}} (a_n^I, a_n^Q)$$

$$D_{\min} = 2\sqrt{\frac{E_s}{3+\sqrt{3}}}$$

151



152



153

$$P_e \approx \frac{2K}{M} Q\left(d_{\min} \sqrt{\frac{E_b}{N_0}}\right)$$

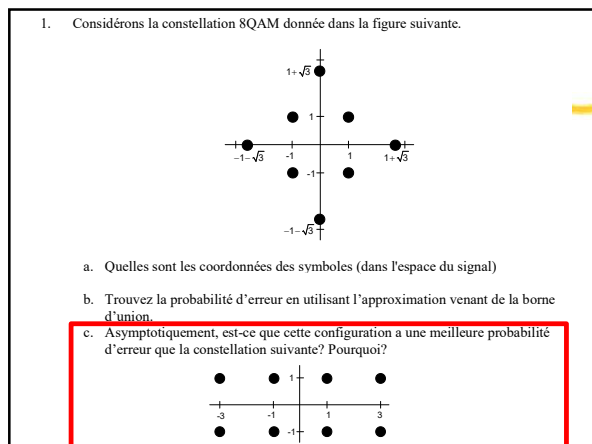
$$D_{\min} = 2\sqrt{\frac{E_s}{3+\sqrt{3}}}$$

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}} = 2\sqrt{\frac{\log_2 8 \cdot E_b}{3+\sqrt{3}} \cdot \frac{1}{2E_b}} = \sqrt{\frac{6}{3+\sqrt{3}}} = \sqrt{1.27}$$

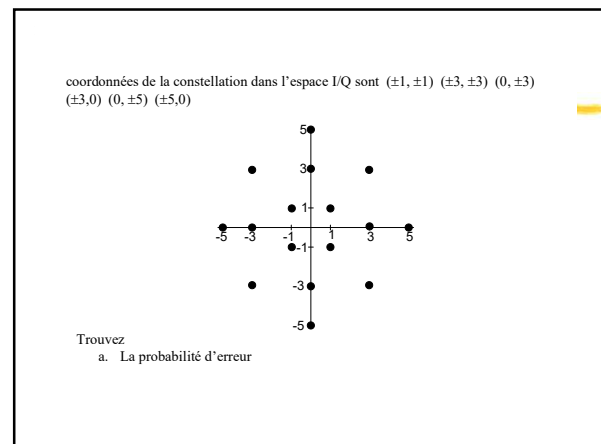
$$D_{\min} = 2\sqrt{E_b}$$

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

154



155



156

GEL7114

Noncoherent detection

157

Topics covered, Part 1



- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4)
- Probability of error & the union bound (course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M -FSK (Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Gray code (Sec. 4.9.4)

158

Part 3

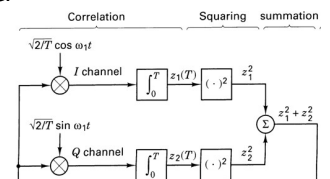
- Modulation
 - PSK
 - FSK
 - ASK
 - QAM
- Scenarios
 - Binary
 - M-ary
 - Coherent detection
 - Noncoherent detection

Error probability will be calculated with non-Gaussian noise

159

Noncoherent detection

- Definition
- Modulations used
 - FSK; DPSK
- Form of receiver



160

PSK or QAM

- Phase modulation or Quadrature amplitude modulation

$$s_i(t) = \sqrt{\frac{2E_b}{T}} \cos(\omega_0 t + \phi_i(t)) \quad i = 1, 2, \dots, M$$

$$\psi_1 = \cos(\omega_0 t) \quad \psi_2 = \sin(\omega_0 t)$$

161

Motivation for BPSK

$$\cos \alpha \cos \beta = .5 \cos(\alpha + \beta) + .5 \cos(\alpha - \beta)$$

$$\cos \omega_0 t \cos(\omega_0 t + \Delta \phi)$$

phase unknown

162

Motivation for BPSK

$$\cos \alpha \cos \beta = .5 \cos(\alpha + \beta) + .5 \cos(\alpha - \beta)$$

$$\cos \omega_0 \cos(\omega_0 + \Delta\phi) = .5 \cos(2\omega_0 + \Delta\phi) + .5 \cos(\Delta\phi)$$

impact of unknown phase

163

Motivation – with data

$$\cos \alpha \cos \beta = .5 \cos(\alpha + \beta) + .5 \cos(\alpha - \beta)$$

$$d \cos \omega_0 \cos(\omega_0 + \Delta\phi) = .5d \cos(2\omega_0 + \Delta\phi) + .5d \cos(\Delta\phi)$$

eliminate with filtering

164

Motivation – with data

$$\cos \alpha \cos \beta = .5 \cos(\alpha + \beta) + .5 \cos(\alpha - \beta)$$

$$d \cos \omega_0 \cos(\omega_0 + \Delta\phi) = .5d \cos(2\omega_0 + \Delta\phi) + .5d \cos(\Delta\phi)$$

$$.5d \cos(\Delta\phi)$$

data easily visible...
when the phase error is zero

$$.5d$$

165

Motivation

$$\cos \alpha \cos \beta = .5 \cos(\alpha + \beta) + .5 \cos(\alpha - \beta)$$

$$\cos \omega_0 \cos(\omega_0 + \Delta\phi) = .5 \cos(2\omega_0 + \Delta\phi) + .5 \cos(\Delta\phi)$$

solutions for arbitrary phase (without a PLL) ???

166

GEL7114

DPSK

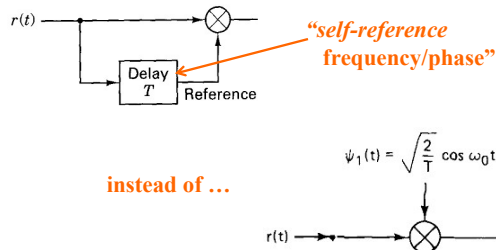
167

DPSK

- Reducing the cost of receiver by avoiding phase tracking
- Phase that's slowly changing
 - Compared to the bit rate
- Information coded in the phase difference from one symbol to another

168

Form of receiver



169

Phase difference

- Data is phase θ

170

Phase difference

- Rappel: $2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$

$$2 \cos[\omega_0 t + \theta_k] \cos[\omega_0 t + \theta_{k-1}]$$

signal received (circled) *self-reference* (circled)

$$= \cos[\omega_0 t + \theta_k + \omega_0 t + \theta_{k-1}] + \cos[\omega_0 t + \theta_{k-1} - \omega_0 t - \theta_k]$$

171

Phase difference

- Rappel: $2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$

$$\begin{aligned} 2 \cos[\omega_0 t + \theta_k] \cos[\omega_0 t + \theta_{k-1}] &= \cos[\omega_0 t + \theta_k + \omega_0 t + \theta_{k-1}] + \cos[\omega_0 t + \theta_{k-1} - \omega_0 t - \theta_k] \\ &= \cos[2\omega_0 t + (\theta_k + \theta_{k-1})] + \cos(\theta_{k-1} - \theta_k) \end{aligned}$$

double-frequency terms are filtered (arrow pointing to the first term)

172

Phase difference

- Recall: $2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$

$$\begin{aligned} 2 \cos[\omega_0 t + \theta_k] \cos[\omega_0 t + \theta_{k-1}] &= \cos[\omega_0 t + \theta_k + \omega_0 t + \theta_{k-1}] + \cos[\omega_0 t + \theta_{k-1} - \omega_0 t - \theta_k] \\ &= \cos[2\omega_0 t + (\theta_k + \theta_{k-1})] + \cos(\theta_{k-1} - \theta_k) \\ &= \cos(\theta_{k-1} - \theta_k) \end{aligned}$$

information (arrow pointing to the final result)
 logical 0: phase change
 logical 1: no phase change

173

Phase difference

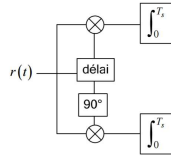
- Noncoherent case

Phase that's slowly changing
 Compared to the bit rate

$$\begin{aligned} 2 \cos[\omega_0 t + \theta_k + \phi] \cos[\omega_0 t + \theta_{k-1} + \phi] &= \cos[\omega_0 t + \theta_k + \phi + \omega_0 t + \theta_{k-1} + \phi] + \cos[\omega_0 t + \theta_{k-1} + \phi - \omega_0 t - \theta_k - \phi] \\ &= \cos[2\omega_0 t + (\theta_k + \theta_{k-1}) + 2\phi] + \cos(\theta_{k-1} - \theta_k) \\ &= \cos(\theta_{k-1} - \theta_k) \end{aligned}$$

174

Supposons que nous avons un système à taux binaire de 1 kb/s et nous exploitons une modulation différentielle de la phase, avec le récepteur non cohérent suivant.



- A. (10 points) Pour envoyer l'information avec modulation DQPSK, quel est le délai du récepteur?
- B. (5 points) Supposons que le canal a une phase qui dérive lentement. La phase reste constante pendant une période de ~2ms. Est-ce que nous pouvons utiliser 16 DPSK pour envoyer l'information à 1 kb/s? Justifiez votre réponse.

175

Pre-coding

- Information in the phase difference
- $m(k)$ symbol for interval kT_s
- $c(k)$ pre-coded symbol for interval kT_s
- Two types of pre-coding

$$c(k) = \overline{c(k-1)} \oplus m(k)$$

$$c(k) = c(k-1) \oplus m(k)$$

176

Transmission

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	π	0	π	π	π	0	π	π

$$c(k) = \overline{c(k-1)} \oplus m(k)$$

177

Transmission $k=1$

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	0	0	π	π	π	0	π	π

$$c(k) = \overline{c(k-1)} \oplus m(k) = \overline{1} \oplus 1 = \overline{0} = 1$$

178

Transmission $k=2$

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	0	0	π	π	π	0	π	π

$$c(k) = \overline{c(k-1)} \oplus m(k) = \overline{1} \oplus 1 = \overline{0} = 1$$

179

Transmission $k=3$

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	0	0	π	π	π	0	π	π

$$c(k) = \overline{c(k-1)} \oplus m(k) = \overline{1} \oplus 0 = \overline{1} = 0$$

180

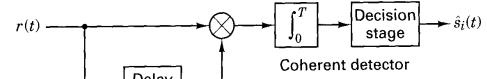
Transmission $k=4$

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	0	0	π	π	π	0	π	π

$$c(k) = \overline{c(k-1)} \oplus m(k) = \overline{0} \oplus 1 = 1 = 0$$

181

Form of receiver



Detected message, $\hat{m}(k)$ 1 1 0 1 0 1 1 0 0 1

phase unknown, noncoherent detection

182

Reception

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	0	0	π	π	π	0	π	π
Detected message, $\hat{m}(k)$		1	1	0	1	0	1	1	0	0	1

$$\hat{m}(k) = \cos[\theta_k - \theta_{k-1}]$$

183

Reception $k=1$

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$		π	π	π	0	0	π	π	π	0	π
Detected message, $\hat{m}(k)$		1	1	1	1	1	1	1	1	1	1

$$\hat{m}(k) = \cos[\theta_k - \theta_{k-1}] = \cos[\pi - \pi] = 1 = \hat{m}(1)$$

184

Reception $k=2$

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	0	0	π	π	π	0	π	π
Detected message, $\hat{m}(k)$		1	1	1	1	1	1	1	1	1	1

$$\hat{m}(k) = \cos[\theta_k - \theta_{k-1}] = \cos[\pi - \pi] = 1 = \hat{m}(2)$$

185

Reception $k=3$

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	0	0	π	π	π	0	π	π
Detected message, $\hat{m}(k)$		1	1	0	1	0	1	1	0	0	1

$$\hat{m}(k) = \cos[\theta_k - \theta_{k-1}] = \cos[\pi - 0] = -1 = \hat{m}(3)$$

186

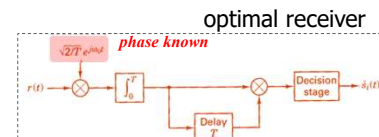
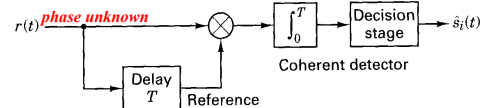
Reception $k=4$

Sample index, k	0	1	2	3	4	5	6	7	8	9	10
Information message, $m(k)$		1	1	0	1	0	1	1	0	0	1
Differentially encoded message (first bit arbitrary), $c(k)$	1	1	1	0	0	1	1	1	0	1	1
Corresponding phase shift, $\theta(k)$	π	π	π	π	0	0	π	π	π	0	π
Detected message, $\hat{m}(k)$		1	1	0	1	0	1	1	0	0	1

$$\hat{m}(k) = \cos[\theta_k - \theta_{k-1}] = \cos[0 - 0] = 1 = m(4)$$

187

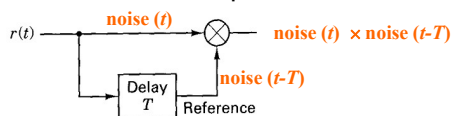
Form of receiver



188

Calculating P_e

- Gaussian noise is multiplied



- Result is a non-Gaussian noise

189

DPSK

- BER for noncoherent receiver
 - A little ($\sim 1\text{dB}$) worse than BPSK/QPSK

$$P_e = \frac{1}{2} e^{-E_b/N_0} \quad Q(x) \approx \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$$

- BER for optimal receiver

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]$$

190

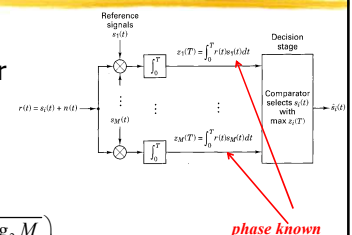
GEL7114

M -FSK noncoherent

191

MFSK coherent receiver

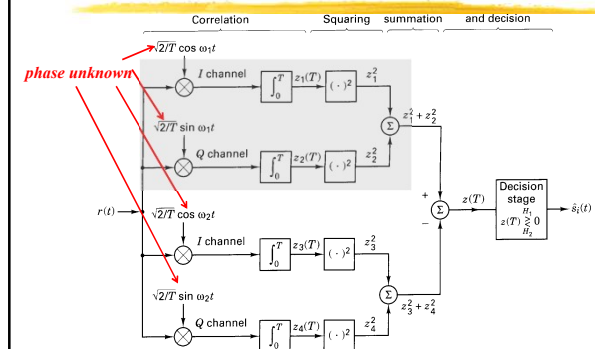
- Form of receiver



$$P_e(M) = (M-1)Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$

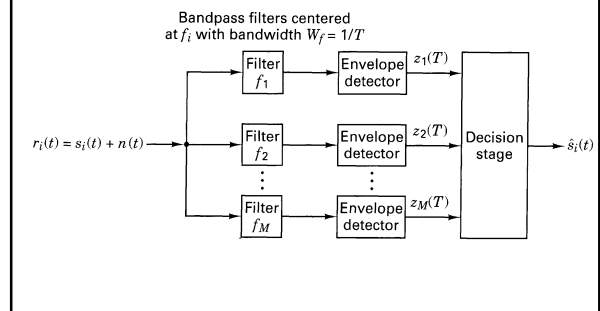
192

MFSK – noncoherent



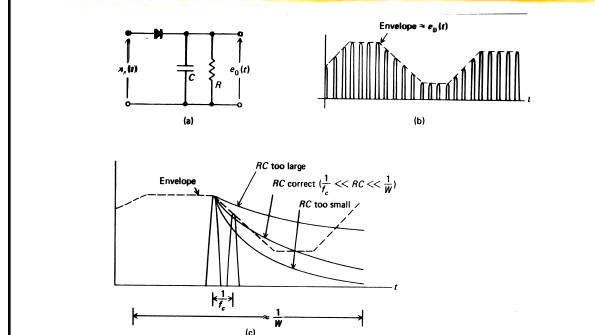
193

MFSK – noncoherent



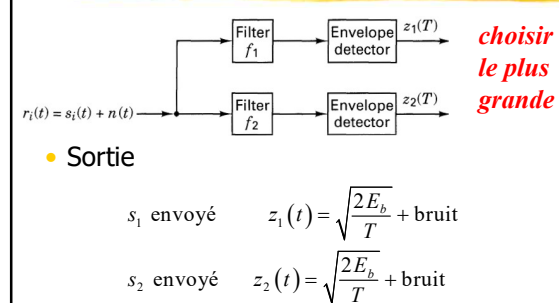
194

Envelope detector



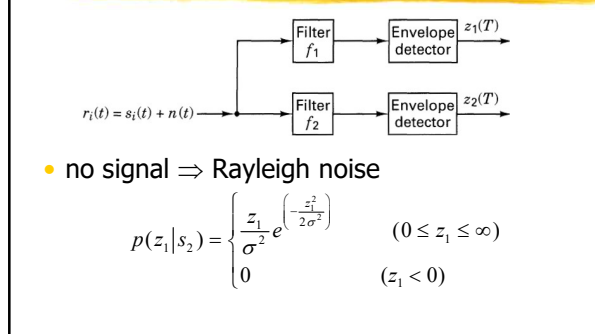
195

Envelope detector



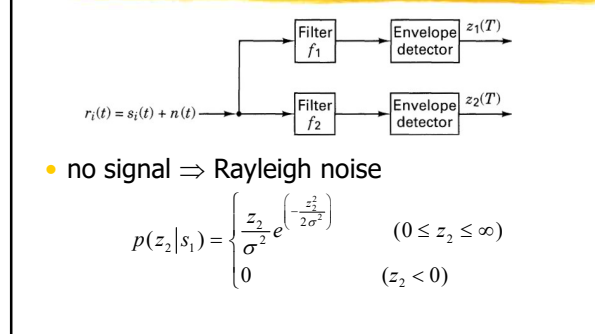
196

Noise in envelope detector



197

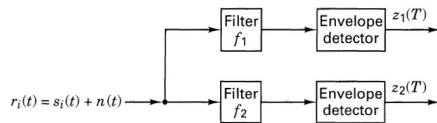
Noise in envelope detector



198

Noise in envelope detector

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos \theta} d\theta$$

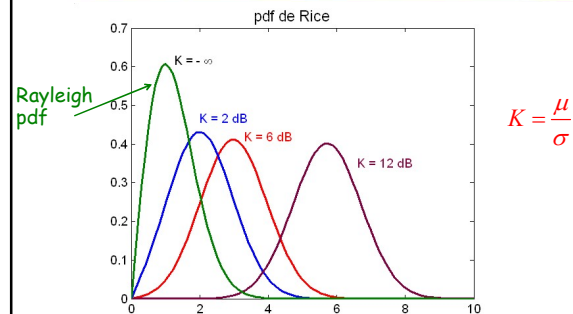


- with signal $\Rightarrow$ Rician noise

$$p(z_1|s_1) = \begin{cases} \frac{z_1}{\sigma^2} e^{-\left(\frac{z_1^2 + \mu^2}{2\sigma^2}\right)} I_0\left(\frac{\mu z_1}{\sigma^2}\right) & 0 \leq z_1 < \infty \\ 0 & z_1 < 0 \end{cases} \quad \mu = \sqrt{\frac{2E_b}{T}}$$

199

Rician PDF



200

Conditional densities

- $\Pr(z_1|s_1)$ Rician with $\mu = \sqrt{\frac{2E_b}{T}}$
- $\Pr(z_1|s_2)$ Rayleigh
- $\Pr(z_2|s_1)$ Rayleigh
- $\Pr(z_2|s_2)$ Rician with $\mu = \sqrt{\frac{2E_b}{T}}$

201

BER

$$\begin{aligned} P_e &= \frac{1}{2} \Pr(z_1 < z_2 | s_1) + \frac{1}{2} \Pr(z_1 > z_2 | s_2) \\ &= \Pr(z_1 > z_2 | s_2) \\ &= \int_0^\infty p(z_2 | s_2) \Pr(z_1 > z_2 | s_2, z_2) dz_2 \\ &= \int_0^\infty p(z_2 | s_2) \left[\int_{z_2}^\infty p(z_1 | s_2) dz_1 \right] dz_2 \end{aligned}$$

202

BER (2)

$$\begin{aligned} P_e &= \int_0^\infty p(z_2 | s_2) \left[\int_{z_2}^\infty p(z_1 | s_2) dz_1 \right] dz_2 \\ &= \int_0^\infty \text{Rice} \left[\int_{z_2}^\infty \text{Rayleigh} dz_1 \right] dz_2 \\ &= \int_0^\infty \frac{z_2}{\sigma^2} e^{-\left(\frac{z_2^2 + \mu^2}{2\sigma^2}\right)} I_0\left(\frac{\mu z_2}{\sigma^2}\right) \left[\int_{z_2}^\infty \frac{z_1}{\sigma^2} e^{-\left(\frac{z_1^2}{2\sigma^2}\right)} dz_1 \right] dz_2 \\ &= \frac{1}{2} e^{-\mu^2/4\sigma^2} = \frac{1}{2} e^{-E_b/2N_0} \end{aligned}$$

203

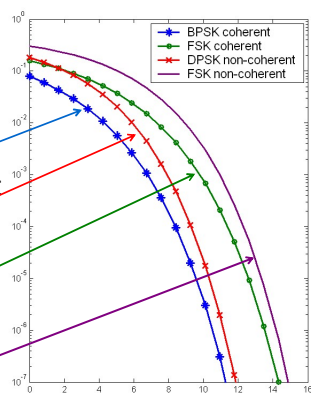
Noncoherent FSK

- BER for noncoherent receiver
 - A little (~ 1 dB) worse than coherent MFSK
 - noncoherent $P_e = \frac{1}{2} e^{-E_b/2N_0}$
 - coherent $P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

204

Coherent vs. noncoherent

- BPSK coherent
 $Q(\sqrt{2E_b/N_0})$
- DPSK noncoherent
 $P_e = .5e^{-E_b/N_0}$
- BFSK coherent
 $Q(\sqrt{E_b/N_0})$
- BFSK noncoherent
 $P_e = .5e^{-E_b/2N_0}$



205

GEL7114

Bit error probability
vs.
Symbol error probability

206

Topics covered, Part 1

- Modulation formats (Sec. 4.2)
- Decision regions for Coherent detection (Sec. 4.3, 4.4)
- Probability of error & the union bound
(course notes, Sec. 4.7.1, 4.7.3, 4.8, 4.9)
- Noncoherent detection & Probability of error for DPSK and M -FSK
(Sec. 4.5.1-4.5.3, 4.7.4-4.7.6)
- Gray code (Sec. 4.9.4)



207

P_b vs. P_s

- Symbol error probability is what we have discussed so far (P_e)
- For bit error probability we have to consider how **bits** errors occur ...

208

Orthogonal modulation

- Equidistant symbols
- P_s = probability of symbol error

$$P_s = \Pr(s_0 \text{ received in error})$$

$$= \sum_{i=1}^{M-1} p(s_i | s_0) = (M-1) p(s_1 | s_0)$$

$$p(s_i | s_j) = \frac{P_s}{M-1}$$

209

Symbol vs. bit error

- M symbols with $[b_1 \dots b_k \dots b_{\log_2 M}]$
- Focus on k^{th} bit
 - One of the M symbols is the correct symbol
 - $M-1$ symbols are incorrect
 - How many of the $M-1$ incorrect symbols have the same b_k as the correct symbol?
 - $M/2 - 1$

210

$$p(s_i | s_j) = \frac{P_s}{M-1}$$

$$P_s = \underbrace{\left(\frac{M}{2} - 1\right) p(s_1 | s_0)}_{\text{no error } k^{\text{th}} \text{ bit}} + \underbrace{\frac{M}{2} p(s_1 | s_0)}_{k^{\text{th}} \text{ bit in error}}$$

$$= \left(\frac{M}{2} - 1\right) \frac{P_s}{M-1} + P_b$$

211

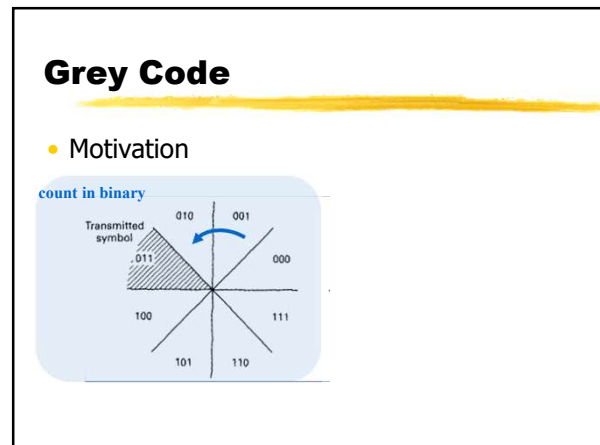
Orthogonal modulation

$$P_b = \frac{M}{2} \frac{P_s}{M-1} = \frac{P_s}{2} \cdot \frac{M}{M-1} \rightarrow \frac{P_s}{2}$$

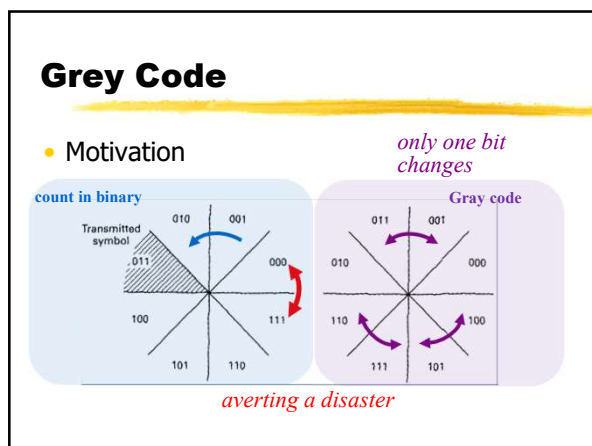
212

- ### Non-orthogonal modulation
- Symbols not necessarily equidistant from each other
 - Error probability dominated by the nearest symbols
 - Ensure that the closest symbols do not induce significant bit changes... [Grey Code](#)

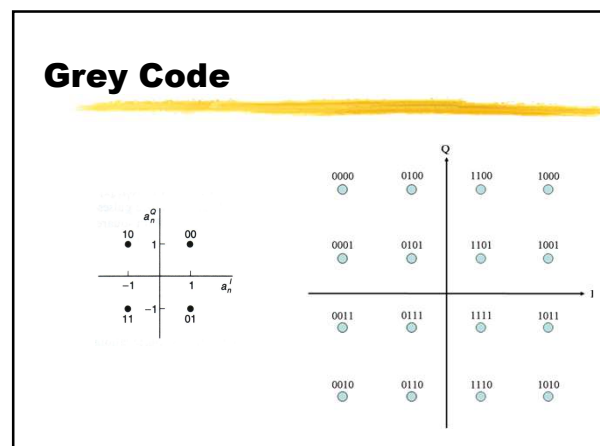
213



214



215



216

Grey Coding

$$P_b \approx \frac{P_s}{\log_2 M} \quad P_s \ll 1$$

217

GEL7114

Minimum spacing for M-FSK

Section 4.5.4

218

Topics covered, part 2

- Minimum spacing for M-FSK (Sec. 4.5.4)
- Spectral efficiency (Sec. 9.7, 9.7.1-9.7.6)
- SNR vs. Eb/N0 (Sec. 3.14, 3.15)
- Channel capacity (Sec. 9.4, 9.4.1)
- Spectral efficiency plan (Sec. 9.6)



219

Orthogonality

- Assumption for MFSK
 - $\psi_i = \cos(\omega_i t)$
 - $\langle \psi_i, \psi_j \rangle = 0 \quad i \neq j$
- Error probability analysis
 - noise alone in the branch without signal, i.e., $\langle \cos(\omega_i t), \cos(\omega_j t) \rangle = 0$
- How to insure orthogonality? What choice of ω_i, ω_j

220

Noncoherent BFSK detection

$$\begin{aligned} & \int_0^T \cos(2\pi f_1 t + \theta) \cos(2\pi f_2 t) dt \\ &= \cos \theta \int_0^T \cos(2\pi f_1 t) \cos(2\pi f_2 t) dt - \sin \theta \int_0^T \sin(2\pi f_1 t) \cos(2\pi f_2 t) dt \\ &= \cos \theta \int_0^T [\cos 2\pi t(f_1 - f_2) + \cos 2\pi t(f_1 + f_2)] dt \\ & \quad - \sin \theta \int_0^T [\sin 2\pi t(f_1 - f_2) + \sin 2\pi t(f_1 + f_2)] dt \end{aligned}$$

221

Noncoherent BFSK detection

$$\begin{aligned} &= \cos \theta \int_0^T [\cos 2\pi t(f_1 - f_2) + \cos 2\pi t(f_1 + f_2)] dt \\ & \quad - \sin \theta \int_0^T [\sin 2\pi t(f_1 - f_2) + \sin 2\pi t(f_1 + f_2)] dt \\ &= \cos \theta \left[\frac{\sin 2\pi t(f_1 - f_2)}{2\pi(f_1 - f_2)} + \frac{\sin 2\pi t(f_1 + f_2)}{2\pi(f_1 + f_2)} \right]_0^T \\ & \quad + \sin \theta \left[\frac{\cos 2\pi t(f_1 - f_2)}{2\pi(f_1 - f_2)} + \frac{\cos 2\pi t(f_1 + f_2)}{2\pi(f_1 + f_2)} \right]_0^T \\ &= \cos \theta \left[\frac{\sin 2\pi T(f_1 - f_2)}{2\pi(f_1 - f_2)} + \frac{\sin 2\pi T(f_1 + f_2)}{2\pi(f_1 + f_2)} \right] \\ & \quad + \sin \theta \left[\frac{\cos 2\pi T(f_1 - f_2) - 1}{2\pi(f_1 - f_2)} + \frac{\cos 2\pi T(f_1 + f_2) - 1}{2\pi(f_1 + f_2)} \right] \end{aligned}$$

222

Noncoherent BFSK detection

$$\begin{aligned}
 &= \cos \theta \left[\frac{\sin 2\pi T(f_1 - f_2)}{2\pi(f_1 - f_2)} + \frac{\sin 2\pi T(f_1 + f_2)}{2\pi(f_1 + f_2)} \right] \\
 &\quad + \sin \theta \left[\frac{\cos 2\pi T(f_1 - f_2) - 1}{2\pi(f_1 - f_2)} + \frac{\cos 2\pi T(f_1 + f_2) - 1}{2\pi(f_1 + f_2)} \right] \\
 &\quad \quad \quad f_1 + f_2 \gg 1 \Rightarrow \frac{1}{2\pi(f_1 + f_2)} \rightarrow 0 \\
 &= \cos \theta \left[\frac{\sin 2\pi T(f_1 - f_2)}{2\pi(f_1 - f_2)} \right] + \sin \theta \left[\frac{\cos 2\pi T(f_1 - f_2) - 1}{2\pi(f_1 - f_2)} \right] \\
 &\quad \quad \quad \text{for orthogonality} \\
 &= \frac{\cos \theta \sin 2\pi T(f_1 - f_2) + \sin \theta [\cos 2\pi T(f_1 - f_2) - 1]}{2\pi(f_1 - f_2)} = 0 \quad \forall \theta
 \end{aligned}$$

223

Noncoherent BFSK detection

$$\begin{aligned}
 &\frac{\cos \theta \sin 2\pi T(f_1 - f_2) + \sin \theta [\cos 2\pi T(f_1 - f_2) - 1]}{2\pi(f_1 - f_2)} = 0 \quad \forall \theta \\
 &\quad \quad \quad \cos = 1 \\
 &\quad \quad \quad \sin = 0 \\
 &\quad \quad \quad \downarrow \\
 &\quad \quad \quad 2\pi T[f_1 - f_2] = 2n\pi \Rightarrow f_1 - f_2 = \frac{n}{T} \quad n \neq 0
 \end{aligned}$$

224

Coherent BFSK detection

$$\begin{aligned}
 &\frac{\cos \theta \sin 2\pi T(f_1 - f_2) + \sin \theta [\cos 2\pi T(f_1 - f_2) - 1]}{2\pi(f_1 - f_2)} = 0 \quad \forall \theta = 0 \\
 &\quad \quad \quad \frac{\sin 2\pi T(f_1 - f_2)}{2\pi(f_1 - f_2)} = 0 \\
 &\quad \quad \quad 2\pi T[f_1 - f_2] = n\pi \Rightarrow f_1 - f_2 = \frac{n}{2T} \quad n \neq 0
 \end{aligned}$$

225

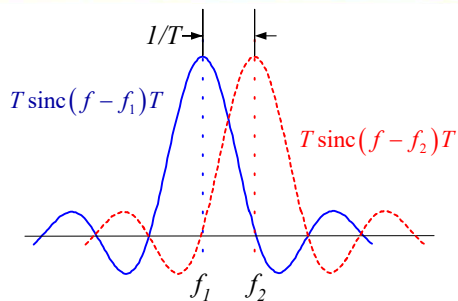
BFSK Detection

	<i>minimal separation</i>
<i>noncoherent</i>	<i>noncoherent</i>
$f_1 - f_2 = \frac{n}{T} \quad n \neq 0$	$f_1 - f_2 = \frac{1}{T}$
<i>coherent</i>	<i>coherent</i>
$f_1 - f_2 = \frac{n}{2T} \quad n \neq 0$	$f_1 - f_2 = \frac{1}{2T}$

226

rectangular pulse in time

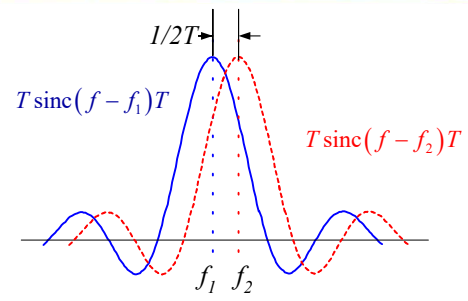
Noncoherent FSK



227

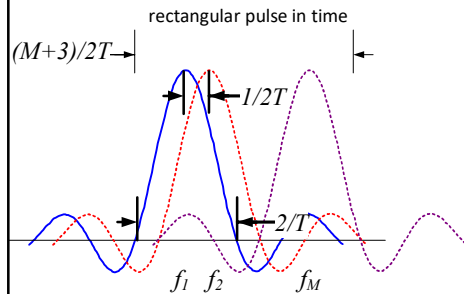
rectangular pulse in time

Coherent FSK



228

Coherent MFSK



229

GEL7114

Spectral efficiency
Sections 9.7, 9.7.1-9.7.6

230

Topics covered, part 2

- Minimum spacing for M-FSK (Sec. 4.5.4)
- Spectral efficiency (Sec. 9.7, 9.7.1-9.7.6)
- SNR vs. E_b/N_0 (Sec. 3.14, 3.15)
- Channel capacity (Sec. 9.4, 9.4.1)
- Spectral efficiency plan (Sec. 9.6)



231

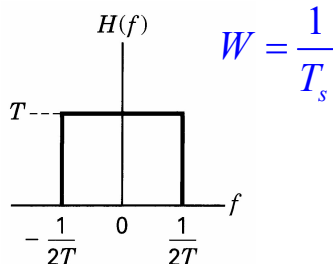
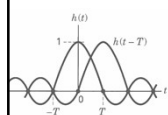
BW Efficiency

- Definition $\eta = \frac{R}{W} = \frac{1}{T_b} \frac{1}{W}$ $T_s = T_b \log_2 M$
- used to compare modulations
- assumes that the signal occupies the smallest possible BW

232

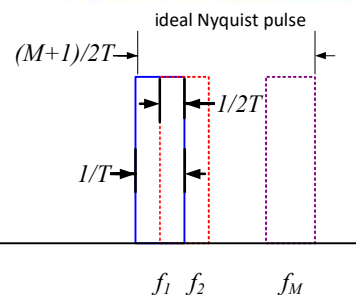
Most bandwidth efficient Nyquist pulse

- $v(t) = \text{Sinc}(t/T)$



233

Coherent MFSK



234

BW efficiency

- Definition $\eta = \frac{R}{W} = \frac{1}{T_b} \frac{1}{W}$ $T_s = T_b \log_2 M$

Coherent MFSK

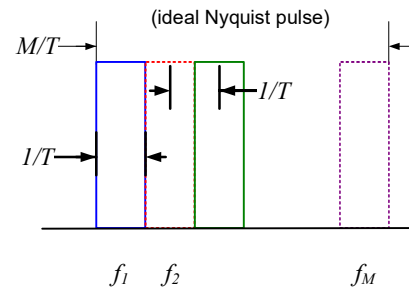
- Each symbol sent on an independent frequency range

$$\Rightarrow W = (M+1)/2T_s$$

$$\eta_{\text{MFSK}} = \frac{2 \log_2 M}{M+1} \text{ b/s/Hz}$$

235

Noncoherent MFSK



236

BW efficiency

- Definition $\eta = \frac{R}{W} = \frac{1}{T_b} \frac{1}{W}$ $T_s = T_b \log_2 M$

MFSK noncoherent

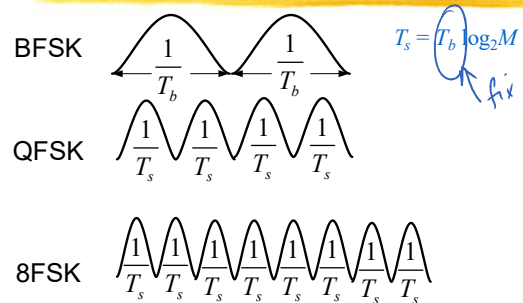
- Each symbol sent on independent frequency ranges

$$\Rightarrow W = M/T_s$$

$$\eta_{\text{MFSK}} = \frac{\log_2 M}{M} \text{ b/s/Hz}$$

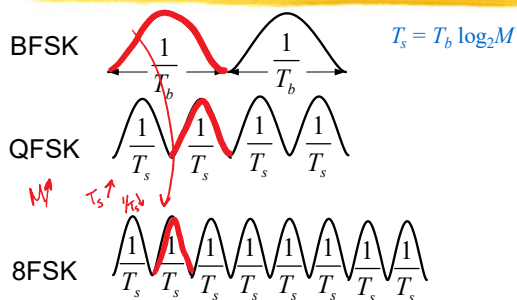
237

BW efficiency - MFSK



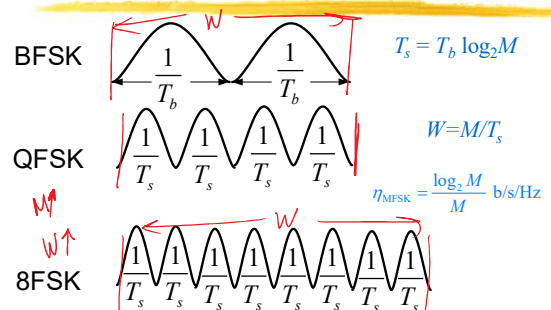
238

BW efficiency - MFSK



239

BW efficiency - MFSK



240

GEL7114

Spectral Efficiency of MPSK

Sections 9.7, 9.7.1-9.7.6

241

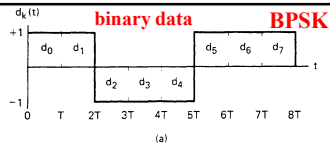
Topics covered, part 2



- Minimum spacing for M-FSK (Sec. 4.5.4)
- Spectral efficiency (Sec. 9.7, 9.7.1-9.7.6)
- SNR vs. E_b/N_0 (Sec. 3.14, 3.15)
- Channel capacity (Sec. 9.4, 9.4.1)
- Spectral efficiency plan (Sec. 9.6)

242

BPSK

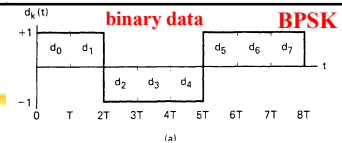


- $R_I = 1/T$
 - $BW_{BPSK} = 1/T$

243

QPSK

same bit rate



- $R_I = 1/2T$
 - $BW_I = 1/2T$
- $R_Q = 1/2T$
 - $BW_Q = 1/2T$
- $R_{QPSK} = 1/T$
 - $BW_{QPSK} = 1/2T$

244

QPSK vs. BPSK

- Bandwidth
 - BPSK: $BW = 1/T$ (ideal pulse)
 - QPSK: $BW = 1/2T$ (ideal pulse)

- Very similar performance

$$P_e(\text{QPSK}) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

twice the P_e of BPSK

245

BW efficiency

- Definition $\eta = \frac{R_b}{BW}$
- BPSK
 - $BW = 1/T_b$ (ideal pulse)
 - $\Rightarrow \eta = 1 \text{ b/s/Hz}$
- QPSK
 - $BW = 1/2T_b$ (ideal pulse)
 - $\Rightarrow \eta = 2 \text{ b/s/Hz}$

246

MPSK – Bandwidth

- Always cos & sin branches
 - same bandwidth covered in each branch
 - time of a symbol = $\log_2 M \times T_b$
- Bandwidth efficiency
 - $BW = 1/T_s = 1/T_b \log_2 M$ (ideal pulse)
 - ➔ $\eta = \log_2 M$ b/s/Hz

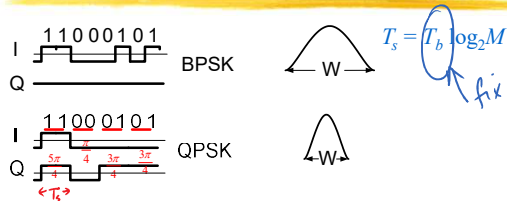
247

BW efficiency – MPSK



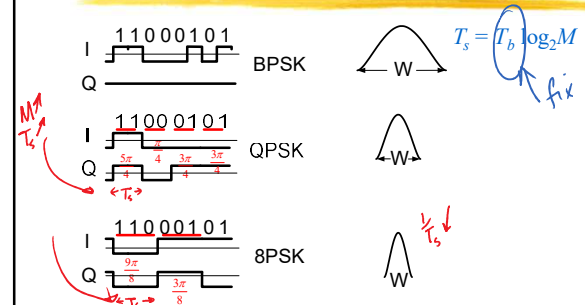
248

BW efficiency – MPSK



249

BW efficiency – MPSK



250

MPSK – P_e et η

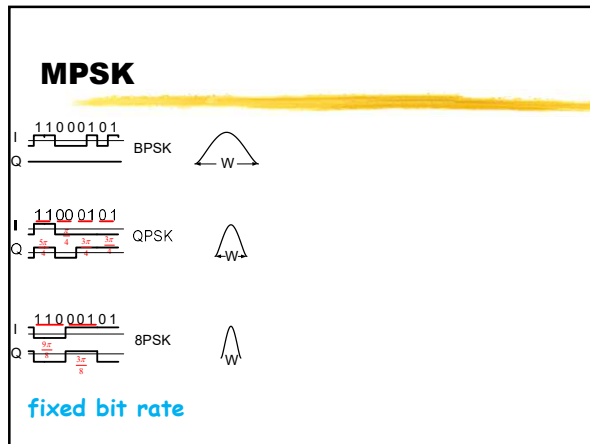
- Efficiency
 - $\eta = \log_2 M$ b/s/Hz
- Error probability
 - Minimal distance $D_{min} = 2\sqrt{E_s} \sin \frac{\pi}{M}$
 - Neighbors at the minimal distance $K=M$
 - $P_e(M) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}} \sin \frac{\pi}{M} \sqrt{\log_2 M}\right)$ loss compared to QPSK

251

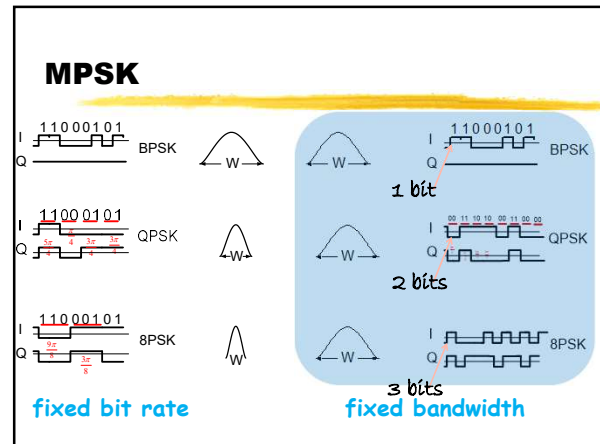
MPSK : $M=8,16$

- Pour $M=8$ $\eta = 3$ b/s/Hz
 - loss compared to QPSK
 - $10\log_{10}\left(\sin \frac{\pi}{8} \sqrt{\log_2 8}\right)^2 = 20\log_{10}[.66] = 3.58\text{dB}$
- Pour $M=16$ $\eta = 4$ b/s/Hz
 - loss compared to QPSK
 - $10\log_{10}\left(\sin \frac{\pi}{16} \sqrt{\log_2 16}\right)^2 = 20\log_{10}[.39] = 8.18\text{dB}$

252



253



254

GEL7114

Contrast: FSK/QAM/PSK

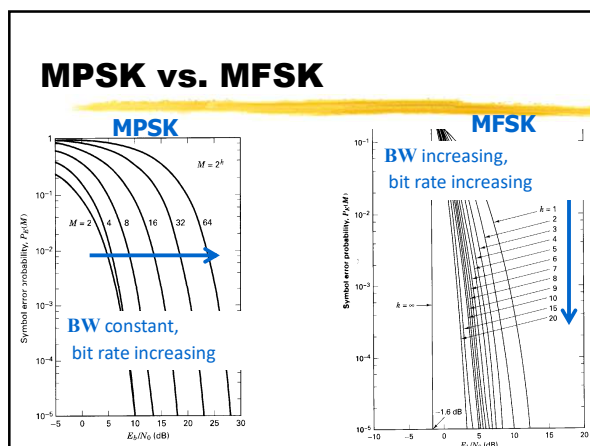
Sections 9.7, 9.7.1-9.7.6

255

Topics covered, part 2

- Minimum spacing for M-FSK (Sec. 4.5.4)
- Spectral efficiency (Sec. 9.7, 9.7.1-9.7.6)
- SNR vs. E_b/N_0 (Sec. 3.14, 3.15)
- Channel capacity (Sec. 9.4, 9.4.1)
- Spectral efficiency plan (Sec. 9.6)

256



257

MFSK vs. MPSK

M	k	R (bit/s)	R_s (syb/s)	MPSK Minimum Bandwidth (Hz)	MPSK R/W	MPSK E_b/N_0 (dB) $P_B = 10^{-5}$	Noncoherent Orthog MFSK Min Bandwidth (Hz)	MFSK R/W	MFSK E_b/N_0 (dB) $P_B = 10^{-5}$
2	1	9600	9600	9600	1	9.6	19,200	1/2	13.4
4	2	9600	4800	4800	2	9.6	19,200	1/2	10.6
8	3	9600	3200	3200	3	13.0	25,600	3/8	9.1
16	4	9600	2400	2400	4	17.5	38,400	1/4	8.1
32	5	9600	1920	1920	5	22.4	61,440	5/32	7.4

258

MFSK vs. MPSK

M	k	R (bit/s)	R_s (sym/s)	MPSK Minimum Bandwidth (Hz)	MPSK E_b/N_0 (dB) $P_B = 10^{-5}$	Noncoherent Orthog MFSK Min Bandwidth (Hz)	MFSK R/W	MFSK E_b/N_0 (dB) $P_B = 10^{-5}$
2	1	9600	9600	9600	1	19,200	1/2	13.4
4	2	9600	4800	4800	2	19,200	1/2	10.6
8	3	9600	3200	3200	3	25,600	3/8	9.1
16	4	9600	2400	2400	4	38,400	1/4	8.1
32	5	9600	1920	1920	5	61,440	5/32	7.4

$$\eta_{\text{MPSK}} = \log_2 M \text{ b/s/Hz}$$

$$\eta_{\text{MFSK}} = \frac{\log_2 M}{M} \text{ b/s/Hz}$$

259

MFSK vs. MPSK

M	k	R (bit/s)	R_s (sym/s)	MPSK Minimum Bandwidth (Hz)	MPSK E_b/N_0 (dB) $P_B = 10^{-5}$	Noncoherent Orthog MFSK Min Bandwidth (Hz)	MFSK R/W	MFSK E_b/N_0 (dB) $P_B = 10^{-5}$
2	1	9600	9600	9600	1	19,200	1/2	13.4
4	2	9600	4800	4800	2	19,200	1/2	10.6
8	3	9600	3200	3200	3	25,600	3/8	9.1
16	4	9600	2400	2400	4	38,400	1/4	8.1
32	5	9600	1920	1920	5	61,440	5/32	7.4

$$P_e (\text{MPSK}) \approx 2Q \left(\sqrt{\frac{2E_b \log_2 M}{N_0}} \sin \frac{\pi}{M} \right)$$

$$P_e (\text{MFSK}_{\text{non-coh}}) = \frac{1}{2} e^{-E_b/2N_0}$$

260

MPSK vs. MFSK

- Power limited system**
 - System with little available power (E_b/N_0)
 - Solution: MFSK
- BW limited system**
 - System with little available bandwidth
 - Solution: MPSK

261

Relative performance

spectral efficiency

power efficiency

$$\eta_{\text{MFSK}} = \frac{\log_2 M}{M} \text{ b/s/Hz}$$

$$P_e (\text{MFSK}_{\text{non-coh}}) = \frac{1}{2} e^{-E_b/2N_0}$$

$$\eta_{\text{MPSK}} = \log_2 M \text{ b/s/Hz}$$

$$P_e (\text{MPSK}) \approx 2Q \left(\sqrt{\frac{E_b}{N_0}} 2 \sin^2 \frac{\pi}{M} \log_2 M \right)$$

$$\eta_{\text{MQAM}} = \log_2 M \text{ b/s/Hz}$$

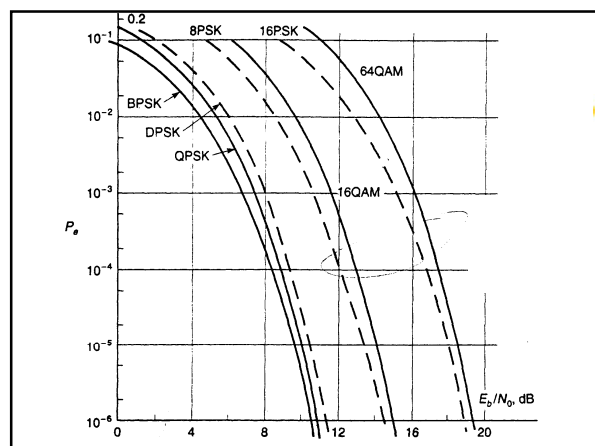
$$P_e (\text{MQAM}) \approx \frac{2K}{M} Q \left(\sqrt{\frac{E_b}{N_0}} \frac{3 \log_2 M}{M-1} \right)$$

262

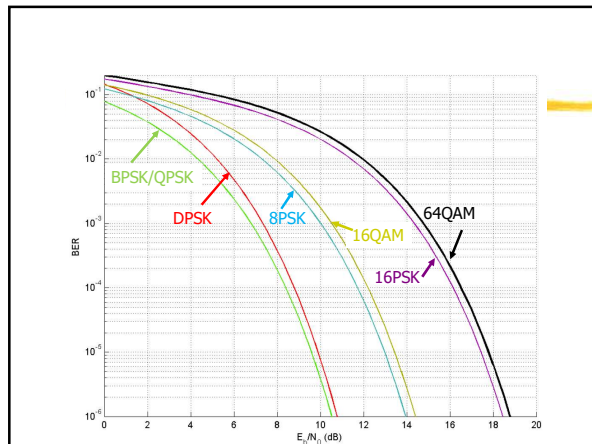
MPSK vs. MQAM

Modulation	Power efficiency (loss asympt.)	Spectral efficiency
BPSK	0 dB	1 b/s/Hz
QPSK	0 dB	2 b/s/Hz
8PSK	3.6 dB	3 b/s/Hz
16PSK	8.2 dB	4 b/s/Hz
16QAM	4 dB	4 b/s/Hz
64QAM	8.4 dB	6 b/s/Hz

263



264



265

Dedicated links

TABLE 7.1 Modem Milestones

Year	Model	Speed (bits/s)	Bandwidth (Hz)	Modulation	R/W (bits/s/Hz)
1962	Bell 201	2,400	1200	4-PSK	2
1967	Milgo 4400/48	4,800	1600	8-PSK	3
1971	Codex 9600C	9,600	2400	16-QAM	4
1980	Paradyne MPI4400	14,400	2400	64-QAM	6
1981	Codex SP14.4	14,400	2400	64-QAM	6
1984	Codex 2660	14,400	2400	Trellis-coded QAM	6
1985	Codex 2680	19,200	2400	Trellis-coded QAM	8

266

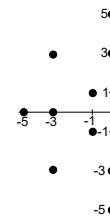
Phone links

TABLE 9.5 Evolution of Dial-Line Telephone Modem Standards

Year	Name	Maximum Bit Rate (bits/s)	Signaling Rate (symbols/s)	Modulation Technique	Signaling Efficiency (bits/symbol)
1984	V.32	9600	2400	2-D Trellis Coded 32-QAM	4
1991	V.32bis	14,400	2400	2-D Trellis Coded 128-QAM	6
1994	V.34	28,800	2400, 2743, 2800, 3000, 3200, 3429	4-D Trellis Coded 960-QAM	~ 9
1996	V.34	33,600	2400, 2743, 2800, 3000, 3200, 3429	4-D Trellis Coded 1664-QAM	~ 10
1998	4.90	downstream: 56,000 upstream: 33,600	8000 as in V.34	PCM* (M-PAM)	7 ~ 10
2000	V.92	downstream: 56,000 upstream: 48,000	8000 8000	PCM* (M-PAM) Trellis Coded PCM*	7 6

267

1. Pour une ligne de téléphone nous utilisons 16QAM *NON*-rectangulaire avec une porteuse à 1700 Hz, un taux de transmission de symbole de $R=2400$ Hz, pour un taux binaire de 9600 b/s. (C'est une version simplifiée de v.29 modem.) Les coordonnées de la constellation dans l'espace I/Q sont $(\pm 1, \pm 1)$ $(\pm 3, \pm 3)$ $(0, \pm 5)$ $(\pm 5, 0)$ $(\pm 5, 0)$ $(0, \pm 5)$



Trouvez

- La probabilité d'erreur
- La bande de fréquence utilisée pour un filtrage idéal de Nyquist
- La bande de fréquence utilisée pour filtrage RC, $r=3$ (largeur de bande en excès)

268

SNR vs. E_b/N_0

269

Topics covered, part 2

- Minimum spacing for M-FSK (Sec. 4.5.4)
- Spectral efficiency (Sec. 9.7, 9.7.1-9.7.6)
- SNR vs. E_b/N_0 (Sec. 3.14, 3.15)
- Channel capacity (Sec. 9.4, 9.4.1)
- Spectral efficiency plan (Sec. 9.6)



270

SNR vs. E_b/N_0

pp. 117, 226
Sklar

- SNR
 - average signal power divided by average noise power
 - S/N
- SNR standardized by bit
 - E_b/N_0

271

E_b vs. S

- Average energy per bit =
average signal power
×
time for a bit

$$E_b = ST_b \quad S = E_b/T_b$$

272

N_0 vs. N

- noise power density =
average noise power
divided by
bandwidth

$$N_0 = N/BW \quad N = N_0 \cdot BW$$

273

SNR vs. E_b/N_0

$$S = E_b/T_b$$

$$N = N_0 \cdot BW$$

$$SNR = \frac{S}{N}$$

$$= \frac{E_b/T_b}{N_0 \cdot BW} = \frac{E_b}{N_0} \frac{1}{T_b} \frac{1}{BW}$$

$$= \frac{E_b}{N_0} R_b \frac{1}{BW} = \frac{E_b}{N_0} \frac{R_b}{BW}$$

$$SNR = \frac{E_b}{N_0} \cdot \eta$$

274

SNR vs. E_b/N_0 for MQAM/MPSK

$$SNR = \frac{E_b}{N_0} \cdot \eta = \frac{E_b}{N_0} \log_2 M \quad M = 2^k$$

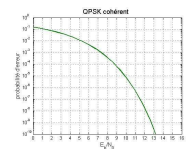
$$\left. \frac{E_b}{N_0} \right|_{db} = SNR|_{db} - \log_2 M|_{db}$$

$$= SNR|_{db} - 10 \log_{10} k|_{db}$$

275

Example 1

- Phone line with 35dB of SNR
- SER for 256 QAM?



276

Example 1

- Phone line with 35dB of SNR
- 256QAM $\Rightarrow$ bit rate $R_b = R_s \times 8 \frac{\text{bits}}{\text{symbole}}$

277

Example 1

- Phone line with 35dB of SNR
- 256QAM $\Rightarrow$ bit rate $R_b = R_s \times 8 \frac{\text{bits}}{\text{symbole}}$
- $SNR = \frac{E_b}{N_0} \frac{R_b}{W} = \frac{E_b}{N_0} \frac{8R_s}{W}$
- ideal Nyquist pulse, $W=R_s$

278

Example 1

- Phone line with 35dB of SNR
 - 256QAM $\Rightarrow$ bit rate $R_b = R_s \times 8 \frac{\text{bits}}{\text{symbole}}$
 - $SNR = \frac{E_b}{N_0} \frac{R_b}{W} = \frac{E_b}{N_0} \frac{8R_s}{W}$
 - ideal Nyquist pulse, $W=R_s$
- $$\frac{E_b}{N_0} = \frac{SNR}{8} \Rightarrow \frac{E_b}{N_0} = 35 - 10\log_{10} 8 = 35 - 9 = 26 \text{ dB}$$

279

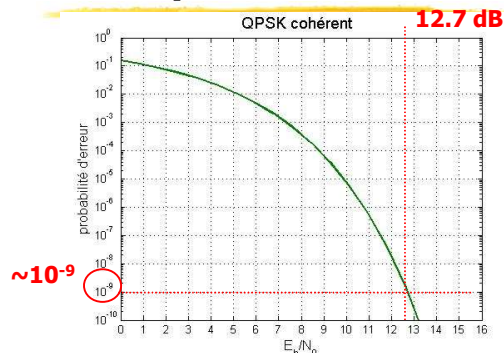
Comparison

26 dB - 13.3 dB = 12.7 dB

- Cost of enlarging the constellation
 - QPSK $d_{\min}(L=2) = \sqrt{\frac{6}{3}} = \sqrt{2}$
 - 256QAM $d_{\min}(L=16) = \sqrt{\frac{6 \times 4}{255}} = \sqrt{\frac{24}{255}} = \sqrt{\frac{12}{255}} \sqrt{2}$
 - loss $-10\log_{10} \frac{12}{255} = 13.3 \text{ dB}$
- loss in dB compared to QPSK

280

loss compared to QPSK



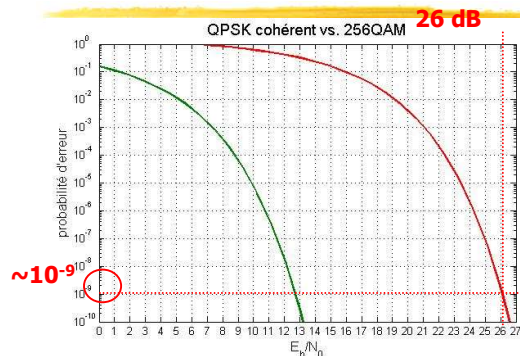
281

Example 1

- Direct calculation
- $$P_e(\text{MQAM}) \approx \frac{2K}{M} Q\left(\sqrt{\frac{E_b}{N_0} \frac{3\log_2 M}{M-1}}\right)$$
- $$\text{Square QAM } P_e = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\log_2 M}{M-1} \frac{E_b}{N_0}}\right)$$

282

Direct calculation



283

Example 2

- Phone line with BW of 2400 Hz
- Bit rate for 256QAM?

284

Example 2

- Phone line with BW of 2400 Hz
- Maximum R_s (symbol rate) (ideal pulse) $R_s = 2400 \frac{\text{symboles}}{\text{sec}}$
- 256QAM $\Rightarrow$ bit rate $R_b = R_s \times 8 \frac{\text{bits}}{\text{symbole}} = 19.2 \text{ kb/s}$

285

Channel capacity

Sections 9.4, 9.4.1, 9.5, 9.7, 9.7.1-9.7.6

286

Topics covered, part 2

- Minimum spacing for M-FSK (Sec. 4.5.4)
- Spectral efficiency (Sec. 9.7, 9.7.1-9.7.6)
- SNR vs. E_b/N_0 (Sec. 3.14, 3.15)
- Channel capacity (Sec. 9.4, 9.4.1)
- Spectral efficiency plan (Sec. 9.6)



287

Complexity ...

- Three important criteria
 - BER vs. E_b/N_0
 - Spectral efficiency $\eta = \frac{R}{W}$
 - Cost/complexity
- New approach to come
 - error corrector codes
 - is there a limit ...

288

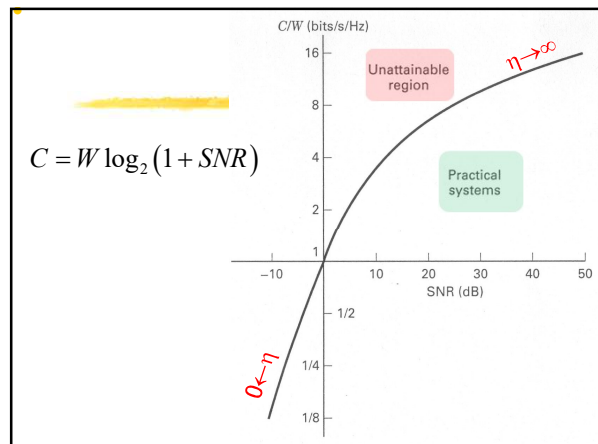
Shannon's Law

- The channel capacity for signal-to-noise SNR and bandwidth W

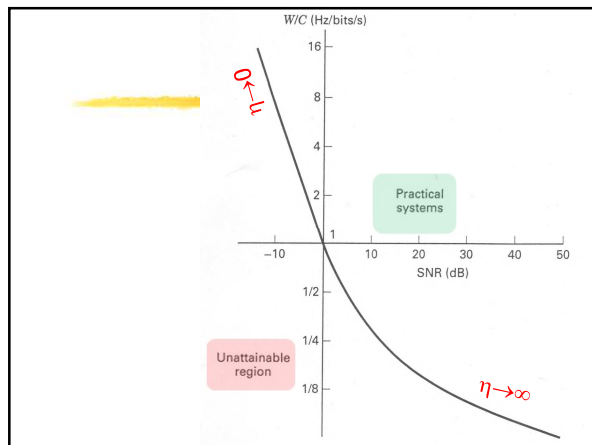
$$C = W \log_2(1 + SNR)$$

- Ability to send C b/s with a P_e as small as you want...

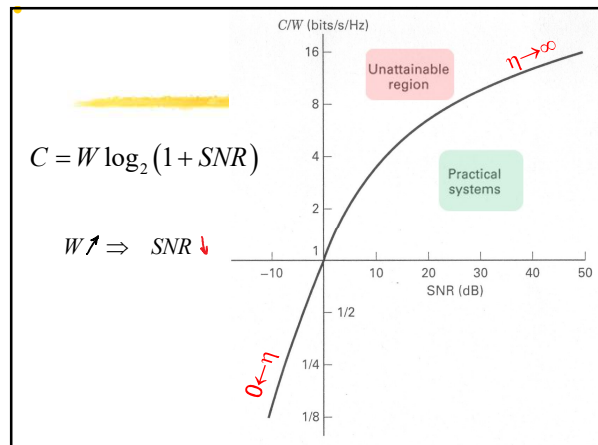
289



290



291



292

border (transmissions are at maximum speed)

Alternative form

$$\left. \frac{C}{W} \right|_{C=R} = \log_2 \left(1 + \frac{S}{WN_0} \right) \Big|_{C=R} = \log_2 \left(1 + \frac{S}{CN_0} \frac{C}{W} \right) \Big|_{C=R}$$

293

border (transmissions are at maximum speed)

Alternative form

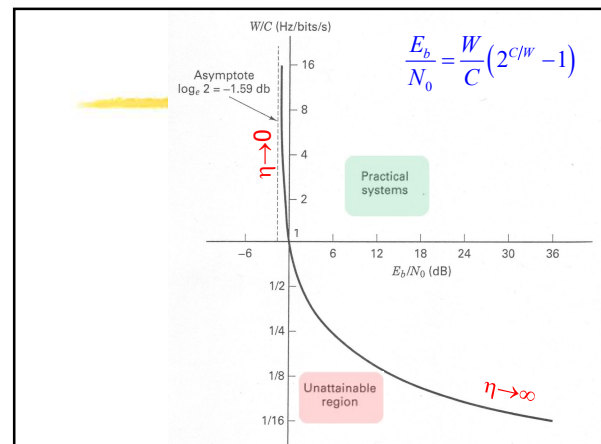
$$\begin{aligned} \left. \frac{C}{W} \right|_{C=R} &= \log_2 \left(1 + \frac{S}{WN_0} \right) \Big|_{C=R} = \log_2 \left(1 + \frac{S}{CN_0} \frac{C}{W} \right) \Big|_{C=R} \\ &= \log_2 \left(1 + \frac{S}{RN_0} \frac{C}{W} \right) \\ &= \log_2 \left(1 + \frac{ST_b}{N_0} \frac{C}{W} \right) = \log_2 \left(1 + \frac{E_b}{N_0} \frac{C}{W} \right) \end{aligned}$$

294

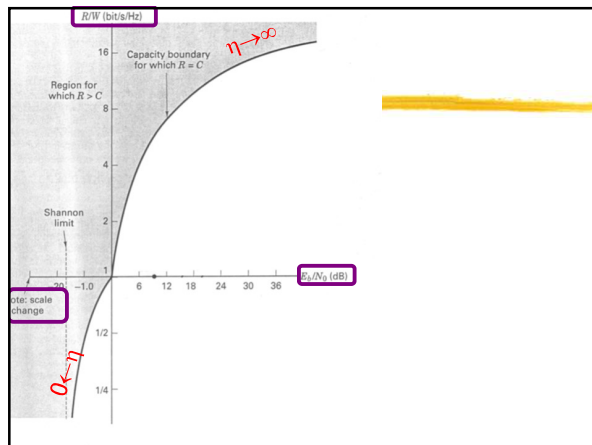
Alternative form

$$\begin{aligned}\left. \frac{C}{W} \right|_{C=R} &= \log_2 \left(1 + \frac{S}{WN_0} \right) \bigg|_{C=R} = \log_2 \left(1 + \frac{S}{CN_0} \frac{C}{W} \right) \bigg|_{C=R} \\ &= \log_2 \left(1 + \frac{S}{RN_0} \frac{C}{W} \right) \\ &= \log_2 \left(1 + \frac{ST_b}{N_0} \frac{C}{W} \right) = \log_2 \left(1 + \frac{E_b}{N_0} \frac{C}{W} \right) \\ \frac{E_b}{N_0} &= \frac{W}{C} (2^{C/W} - 1) \quad \frac{C}{W} \rightarrow 0 \Rightarrow \frac{E_b}{N_0} \rightarrow -1.6 \text{ dB}\end{aligned}$$

295



296



297

GEL7114

Spectral efficiency plane Section 9.5

298

Topics covered, part 2

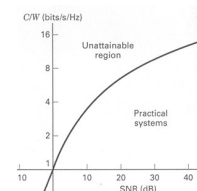
- Minimum spacing for M-FSK (Sec. 4.5.4)
- Spectral efficiency (Sec. 9.7, 9.7.1-9.7.6)
- SNR vs. E_b/N_0 (Sec. 3.14, 3.15)
- Channel capacity (Sec. 9.4, 9.4.1)
- Spectral efficiency plane (Sec. 9.6)



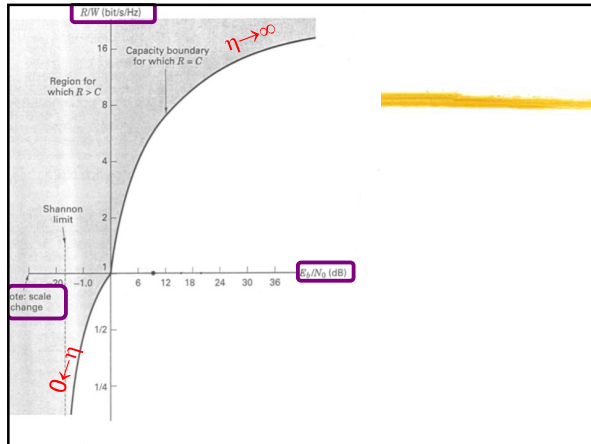
299

Spectral efficiency plane

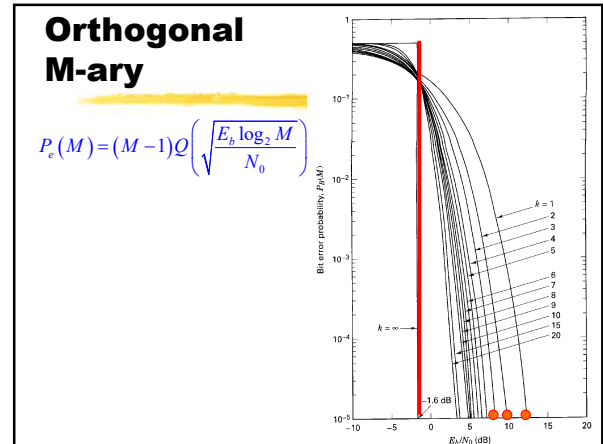
- Visualization of spectral efficiency
- Visualization of the Shannon limit
- SER visualization



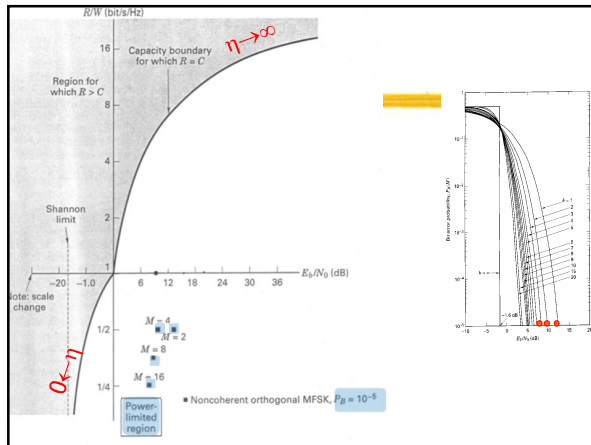
300



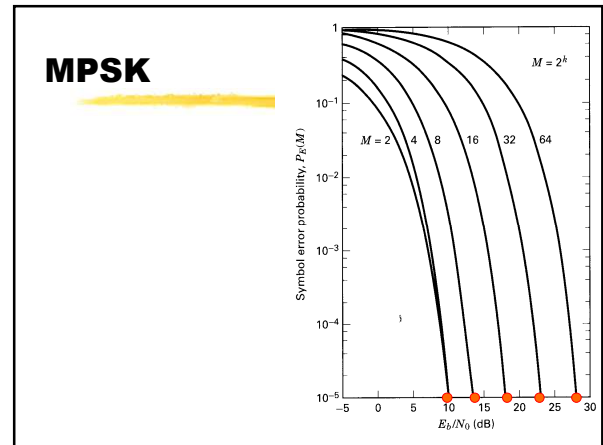
301



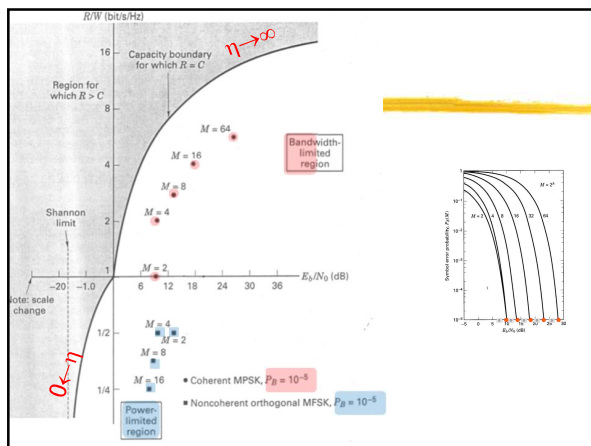
302



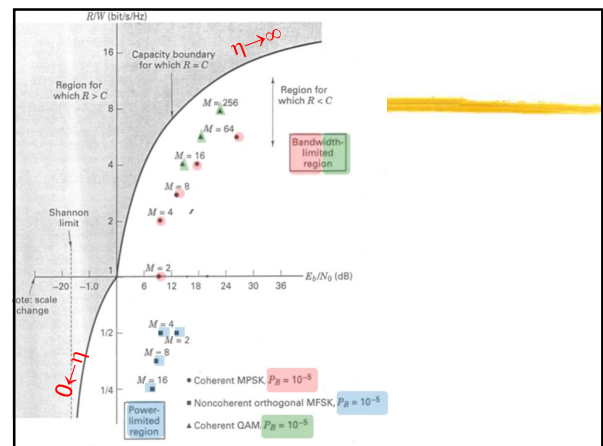
303



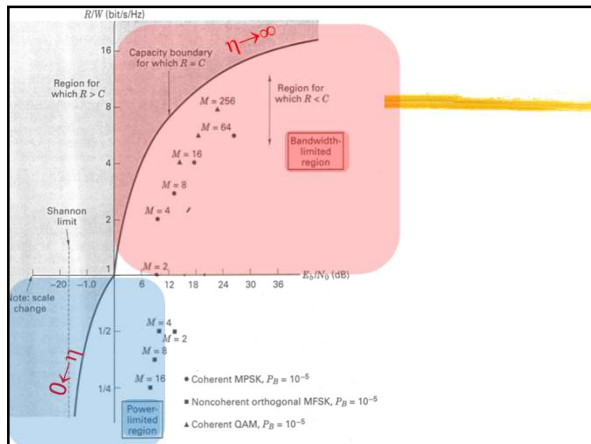
304



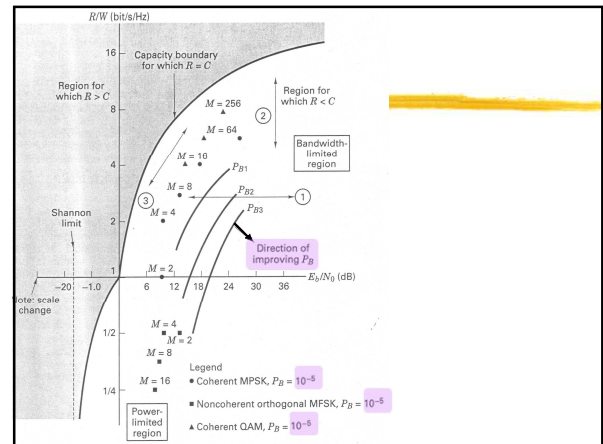
305



306



307



308

2017 midterm exam

Problem 2

Consider the graph of the "Spectral efficiency plane". Find the coordinates of 16 QAM (square) and 8FSK (consistent) for an error probability of 10^{-6} .

309

2011 midterm exam

Problem 2

Consider the graph of the "Spectral efficiency plane".

I have developed a new LAR modulation system which is as spectrally efficient as MPSK. The penalty of 8LAR against QPSK is 5 dB, and the penalty of 16LAR against QPSK is 8 dB.

Find the coordinates of 8LAR and 16LAR for the "spectral efficiency plane" graph.

310

2014 midterm exam

Consider the graph of the "Spectral Efficiency Plane".

Find the coordinates of 3PAM, 4PAM and 8PAM for a probability of error of 10^{-5}

311